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OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. VIII.

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Page 241, line 12, *for* Season *read* Session.

Page 242, *under* Clifford, W. K., F.R.S., *add* line 1, 225—233.

Page 243, *under* Sylvester, J. J., F.R.S., *add* 235.

Page 244, *to* Gauss *add* 202.

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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. VIII.

THIRTEENTH SESSION, 1876–77.

November 9th, 1876.

ANNUAL GENERAL MEETING, held at 22, Albemarle Street.

Prof. H. J. S. SMITH, F.R.S., President, in the Chair.

THE Treasurer (S. Roberts, Esq., M.A.) read his Report, and Prof. A. G. Greenhill, M.A., consented to act as Auditor. The Secretaries' Report was then read and accepted. From the Report of the Secretaries, it appeared that the number of Members had increased from 133 to 135 during the past Session, and that the present number of Life Members is 38.

The Obituary of the Society, during the Session, contained the name of Mr. John Taylor, Benbow Place, Shrewsbury.

The communications made to the Society during the past Session had been as follows:—

"On the Fifteen Young Ladies' Problem, and a general Mathematical Theory of Pure Syntax:" by Prof. Sylvester, F.R.S.

"On the Relation between Bernoulli's Numbers and the Binomial Coefficients:" by Mr. J. Hammond, B.A.

"On Three-Bar Motion in Plane Space:" by Mr. S. Roberts, M.A.

"Values of certain Infinite Products, with an Application to the Summation of the Geometrical Series of the n^{th} order as a Definite Integral:" by Mr. J. W. L. Glaisher, F.R.S.

"On the form of Cam which, acting on a Lever, shall communicate a motion such that the angular-velocity-ratio of the lever and

cam is a given function of the angle described by the latter:" by Major J. R. Campbell.

"The Transformation of Elliptic Functions:" by Prof. Clifford, F.R.S.

"On a System of Algebraical Equations:" by Prof. Cayley, F.R.S.

"On a Problem of Eisenstein's;" "On the Joint-Invariants of two Conics or Quadrics;" and "On the Equation $p\mathrm{D} = \text{constant}$ of a Geodesic Line on an Ellipsoid:" by Prof. H. J. S. Smith, F.R.S.

"On the Solution of certain Partial Differential Equations of the Second Order, having more than two Independent Variables:" by Prof. H. W. Lloyd Tanner, M.A.

"On an Elliptic-Function Identity:" by Mr. J. W. L. Glaisher.

"The Solution of Partial Differential Equations of the Second Order, with any number of Variables when there is a complete First Integral:" by Prof. Tanner.

"Free Motion of a Rigid System in an n-fold Homaloid; expression of the velocities by Abelian functions:" by Prof. Clifford.

"On the Approximate Solution of certain Potential Problems:" by Lord Rayleigh, F.R.S.

"Loci connected with the Rectangular Hyperbola, being inverses with respect to its centre and vertices:" by Prof. Wolstenholme, M.A.

"On Determinants of Alternate Numbers:" by Mr. Spottiswoode, F.R.S.

"On the Transformation of Gauss's Hypergeometric Series into a Continued Fraction:" by Mr. T. Muir, M.A.

"On the Partition of Geometrical Curves:" by Prof. Smith.

"On the Sum of the Products of r different Terms of a Series:" by Mr. Hammond.

"On Pendular Motion:" by Prof. Clifford.

"On the Bicursal Sextic, and the Problem of Three-Bar Motion:" by Prof. Cayley.

"On the Classification of Geometric Algebras:" by Prof. Clifford.

"Note sur un Théorème d'Eisenstein:" by M. C. Hermite.

"The Aspects of Circles on a Plane or on a Sphere:" by Prof. Smith.

"On Correlation in Space:" by Prof. Sturm, of Darmstadt.

"On the Inscription of a Polygon in a Conic Section, subject to the condition that each of its sides shall pass through a given point, by the aid of the generating circle of the conic:" by Mr. S. A. Renshaw.

"On the Representation of Imaginary Quantities by an (n, n) Correspondence:" by Prof. Cayley.

"On the Value of a certain Determinant;" and "On a Method of Solving the Pellian Equation:" by Prof. H. J. S. Smith.

"A general Method of describing Plane Curves of the n^{th} degree by Linkwork:" by Mr. A. B. Kempe, B.A.

"Further Note on the Motion of a Plane under certain Conditions:" by Mr. Roberts.

"A Note on the Equation to the Nodal Cubic:" by Mr. J. J. Walker, M.A.

Shorter communications were also made by Profs. Cayley and H. J. S. Smith.

The same Mathematical Journals had been subscribed for as in the preceding Session; and, in addition to the journals and papers presented to the Society, the "Journal of Education" had been received monthly.

The Proceedings of the Society had been sent to, and the Proceedings of other Societies had been received from, the same persons and Societies as in the previous Session.

The following presents had been made to the Society:—

"Ueber die congruenten Transformationen der bilineären Formen;" "Ueber Schaaren von quadratischen und bilineären Formen;" "Ueber die algebraischen Gleichungen von denen die Theilung der elliptischen Functionen abhängt;" "Ueber quadratische Formen von negativen Determinante;" "Bemerkungen zur Geschichte des Reciprocitätsgesetzes:" from the Author, Dr. Kronecker.

"The Repulsion of Solid Bodies referable to Radiation; a Speculation concerning Molecular Physics:" from the Author, the Rev. A. J. Stevens, M.A.

"On a general Method of producing exact Rectilinear Motion by Linkwork:" from the Author, A. B. Kempe, B.A.

"Ueber die v. Staudt'schen-würfe:" from the Author, Rud. Sturm, in Darmstadt.

"Ueber eine Relation zwischen den Singularitäten einer algebraischen Curve;" "Ueber den Zusammenhang der Flächen:" from the Author, Dr. Felix Klein.

"The Theory of Screws, a study in the Dynamics of a Rigid Body:" from the Author, Dr. R. Stawell Ball, F.R.S.

"Une réforme géométrique, introduction à la géométrie descriptive des cristalloïdes;" "Géométrie hugodomoïdale anhellénique, mais philosophique et architectonique; la question de l'équidomoïde et des cristalloïdes géométriques;" "The Pan-imaginary Theory:" from the Author, le Comte Léopold Hugo.

"Jornal de Sciencias Mathematicas, Physicas, e Naturaes, poblicado sob os auspicios da Academia Real das Sciencias de Lisboa:" from Mr. W. S. W. Vaux.

"Sur une classe de points singuliers de surfaces;" "Note sur les

singularités des courbes planes;" "Sur la théorie des surfaces réciproques:" from the Author, Dr. H. G. Zeuthen.

"Extrait d'une lettre de M. Ch. Hermite à M. L. Königsberger sur le développement des fonctions elliptiques suivant les puissances croissantes de la variable;" "Sur les développements de la fonction $F(x) = sn^a x \,.\, cn^b x \,.\, dn^c x$, où les exposants sont entiers;" "Extrait d'une lettre de M. Ch. Hermite de Paris à M. L. Fuchs de Gottingue, sur quelques équations différentielles linéaires;" "Lettre de M. Ch. Hermite à M. Borchardt sur la fonction de Jacob Bernoulli;" "Extrait d'une lettre de M. Ch. Hermite à M. Borchardt sur la transformation des formes quadratiques ternaires en elles-mêmes;" "Extrait d'une lettre de M. Ch. Hermite à M. Borchardt sur la réduction des formes quadratiques ternaires;" "A property of Bernoulli's Numbers:" from the Author, M. Ch. Hermite.

"Nouvelles propriétés géométriques de la surface de l'ordre qui s'interprètent en optique;" "Démonstration géométrique d'une relation due à M. Laguerre;" "Recherches sur la surface de l'onde;" "Propriétés des diamètres de la surface de l'onde et interprétation physique de ces propriétés:" from the Author, M. Mannheim.

"Nuova teoria delle soluzioni singolari delle equazioni differenziali di primo ordine e secondo grado tra due variabili;" "Sulla teoria delle soluzioni singolari delle equazioni differenziali:" from the Author, Prof. F. Casorati.

"Zur Theorie der algebraischen Flächen;" "Das Problem der Collineation:" from the Author, Dr. Rudolf Sturm, Darmstadt.

"Eine neue relation zwischen den Singularitäten einer algebraischen Curve;" "Ueber lineare Differential-gleichungen;" "Ueber Abelsche Integrale;" "Ueber Riemannsche Flächen:" from Prof. Klein.

"Jahrbuch über die Fortschritte der Mathematik," Sechster Band, Jahrgang 1874, Heften 1, 2; Berlin, 1876.

"Truth in Extremis: a Plea against the Murder of Science by the Gold Poison." By A Searcher. Oxford, 1876.

"Sulle soluzioni singolari delle equazioni alle derivate parziali:" from the Author, Prof. Felice Casorati.

"Proceedings of the Musical Association for the Investigation and Discussion of Subjects connected with the Art and Science of Music," Second Session, 1875–6.

"Elementary Treatise on Elliptic Functions:" from the Author, Prof. Cayley, F.R.S.

Cartes de Visite of Drs. Klein, Kronecker, Zeuthen, and Mr. T. Muir had been received.

The Meeting proceeded to the election of the new Council. The Scrutators (Major J. R. Campbell and Mr. Rawson) having examined the balloting lists, declared the following gentlemen (nominated by the Council) duly elected:—

President—The Right Hon. Lord Rayleigh, F.R.S. Vice-Presidents—C. W. Merrifield, Esq., F.R.S.; Prof. H. J. S. Smith, F.R.S.; Dr. W. Spottiswoode, F.R.S. Treasurer—S. Roberts, Esq., M.A. Hon. Secretaries—M. Jenkins, Esq., M.A.; R. Tucker, Esq., M.A. Other Members—Prof. Cayley, F.R.S.; Prof. Clifford, F.R.S.; T. Cotterill, Esq., M.A.; J. W. L. Glaisher, Esq., F.R.S.; Rev. R. Harley, F.R.S.; Prof. Henrici, F.R.S.; Dr. Hirst, F.R.S.; *A. B. Kempe, Esq., B.A.; *J. J. Walker, Esq., M.A. (The gentlemen marked * were not on the previous Council list.)

Lord Rayleigh, F.R.S., having taken the Chair, briefly returned thanks for the honour done him by the Society.

Prof Greenhill was admitted into the Society. Mr. H. Weston Eve, M.A., formerly Fellow of Trinity College, Cambridge, Head-Master of University College School, London, and Mr. R. F. Davis, B.A., Scholar of Queens' College, Cambridge, were proposed for election.

The following visitors were present:—Messrs. R. F. Davis, W. Barrett Davis, and G. W. von Tunzelmann.

Prof. H. J. S. Smith then delivered his valedictory address, taking for his subject "The Present State and Prospects of some Branches of Pure Mathematics." On the motion of Prof. Cayley, it was agreed unanimously, after the Author's consent had been obtained, that the address should be printed in the Society's Proceedings.

Mr. Harley, Prof. Clifford, and the Chairman made a few remarks on the address.

Mr. J. W. L. Glaisher next gave an account of his paper "On certain Identical Differential Relations."

Mr. Tucker read portions of papers by Dr. Spottiswoode, "On Curves having Four-Point Contact with a triply-infinite Pencil of Curves;" and by Mr. E. B. Elliott, "On some Classes of Multiple Definite Integrals."

The following presents were made to the Society:—

"Monatsbericht," Juli, August, 1876.

"Journal of Education," November, 1876.

"Bulletin de la Société Mathématique de France," Tome iv., Juin, No. 6.

"Bulletin des Sciences Mathématiques et Physiques," Juillet, Août, 1876, Tome xi.; and "Table des Matières et Noms d'Auteurs," Tome x., i^{er} sémestre, 1876.

"Annali di Matematica Pura ed Applicata," Serie ii^{a}, Tomo vii°, fasc. 4°, Ottobre 1876.

"Proceedings of the Royal Society," Vol. xxv., No. 174.

On the Present State and Prospects of some Branches of Pure Mathematics. By H. J. Stephen Smith, Savilian Professor of Geometry in the University of Oxford.

[*Read November 9th*, 1876.]

I have been led to believe that the Society may not be unwilling to allow a certain latitude in the scope of the remarks which they permit their Presidents to address to them upon retiring from the Chair. Relying upon this belief, I propose, on the present occasion, to invite your attention to some considerations relating to the present state of Mathematical Science, with especial reference to its cultivation in this country, and to our own position as representing a great number of those who are interested in its advancement. The subject is so extensive that I am sure you will excuse me if I endeavour to limit it in every way I can. I propose, therefore, to exclude from what I have to say all that relates to Applied Mathematics, and to ask you to confine your attention to questions of Pure Mathematics only. I am well aware how much by this exclusion I restrict the field before me; but the restriction is forced upon me, not only by the limit of time, but by the far narrower limits of my own knowledge. And I cannot help adding that I shall regard it as a fortunate circumstance, if the attention of my successor, when he in his turn is looking round him for a subject for his own Presidential Address, should be attracted by a domain, upon which I must myself decline to enter, but of which he, better perhaps than anyone among us, is fitted to take a clear and comprehensive view.

The restriction which I have mentioned is far from being the only one which I must impose upon myself. I can only presume to offer fragmentary remarks upon great subjects, in the hope that even such casual and hasty notices may not be without their use, if they serve to remind us of the vastness of our science, and yet of its unity; of its unceasing development, rapid at the present time, and promising to be no less rapid in the immediate future; of its marvellous power of assimilating to itself the accessions which each year brings to our knowledge of external nature, while yet it derives strength and vitality from roots which strike far back into the past, so that the organic continuity of its gigantic growth has been preserved throughout.

In every science there is a time and place for general contemplations, as well as for minute investigations. And it is a rule of sound philosophy that neither should be neglected in its proper season. "Itaque alternandæ sunt istæ contemplationes," says Lord Bacon, "et vicissim sumendæ, ut intellectus reddatur simul penetrans et capax."* Per-

* Novum Organon, Lib i., Aph.

haps it is the besetting sin of mathematicians to concentrate the mental vision upon as narrow and definite a field as possible. And there is much to be said in excuse for our indulgence of this tendency. If we are to find anything worth finding in the mines of mathematical research, we must dig deep; and if we want to dig deep, we must, if we are not gifted with Herculean force, confine our efforts to a narrow superficial area. But the tendency is not without its peril. The illustrious mathematician under whose auspices this Society was founded, felt it right in his opening address to warn us against the danger. "Our subject," said Augustus De Morgan, on the 15th Jan. 1865, with a characteristic irony of expression, "Our subject is really rather a wide one. But there are mathematical publications in which it is contracted; and it is often treated as a narrow subject." He cautioned us against falling into "a line which may be useful, but which is still confined and partial"; and, while exhorting us to do our part in the additions to the more rapidly developing "branches of the science," he bid us at the same time take care "not to let any one particular branch overgrow us." It would not have been Augustus de Morgan if he had not added some pointed criticisms upon examinations in general, and on Cambridge examinations in particular, and if he had not cautioned us against any excessive admiration for that part of mathematical ingenuity which devotes itself to the narrowest of all the narrow fields ever chosen by a mathematician, the invention and solution of "ten-minute conundrums."

It is now nearly twelve years since these warnings were given to the infant Society by its first President; and perhaps the time may have arrived when we might put to ourselves the question whether its subsequent history has shown that we have profited by the lessons of that eminent and large-minded teacher. It would ill become me to attempt to answer such a question. I would only venture to express, and that with great diffidence, the double opinion—that, on the one hand, the mathematical world will wholly acquit the Society of having devoted its energies to little or trivial subjects; but that, on the other hand, while it would be universally conceded that the volumes of our Proceedings contain memorable additions to mathematical knowledge, it might be alleged by an "advocatus diaboli" (if such a character should be assumed by some severe critic) that we, in this respect resembling the other mathematicians of our country, have shown, and still continue to show, a certain partiality in favour of one or two great branches of the science, to the comparative neglect and possible disparagement of others. Perhaps it would be well to begin our reply by denying the charge; but, having done so, if we should be advised to urge a second and somewhat contradictory plea, we might with great plausibility rejoin that ours is not a blameable partiality, but a well-grounded prefer-

ence. So great (we might contend) have been the triumphs achieved in recent times by that combination of the newer algebra with the direct contemplation of space which constitutes the modern geometry —so large has been the portion of these triumphs which is due to the genius of a few great English mathematicians—so vast and so inviting has been the field thus thrown open to research,—that we do well to spend our time and our labour upon a country which has, we might say, been "prospected" for us, and in which we know beforehand that we cannot fail to obtain results which will repay our trouble, rather than adventure ourselves into regions where, soon after the first step, we should have no beaten tracks to guide us to the lucky spots, and where the daily earnings of the searcher for mathematical treasure are (at the best) but small, and do not always make a great show even after long years of work. Such regions, however, there are in the domain of pure mathematics, and it cannot be for the interest of science that they should be altogether neglected by the rising generation of English mathematicians.

I propose therefore, in the first instance, to direct your attention to some few of these by us comparatively neglected regions; and foremost among them I must name the Theory of Numbers. Of all branches of mathematical enquiry this is the most remote from practical applications; and yet, more perhaps than any other, it has kindled an extraordinary enthusiasm in the minds of some of the greatest mathematicians. We have the examples of Fermat, Euler, Lagrange, Legendre, and Gauss, of Cauchy, Jacobi, Lejeune Dirichlet, and Eisenstein, without mentioning the names of others who have passed away, and of a few who are still living. But, somehow, the practical genius of the English mathematician has in general given a different direction to his pursuits; and it would sometimes seem as if we in England measured the importance of the subject by what we find of it in our text-books of Algebra, or as if we regarded its enquiries as problems of mere curiosity, without a wider scope, and without direct bearing on other branches of mathematics. I might endeavour to remove this impression—if indeed it exists in the minds of any of those who hear me—by enumerating instances in which the advancement of Algebra and of the Integral Calculus appears to depend on the progress of the arithmetic of whole numbers. But, instead of wearying you with the details which would be necessary to make such an enumeration intelligible, I would rather ask you to listen to what is recorded of the most eminent master of this branch of science. "Gauss," we are told by his biographer, "held Mathematics to be the Queen of the Sciences, and Arithmetic to be the Queen of Mathematics." "She sometimes condescends"—so spoke the great Astronomer and Physicist—"to render services to astronomy and the other natural sciences, but under all circumstances the first

place is her due."* In a more serious mood he wrote, "The higher arithmetic presents us with an inexhaustible storehouse of interesting truths—of truths, too, which are not isolated, but stand in the closest relation to one another, and between which, with each successive advance of the science, we continually discover new and sometimes wholly unexpected points of contact. A great part of the theories of Arithmetic derive an additional charm from the peculiarity that we easily arrive by induction at important propositions, which have the stamp of simplicity upon them, but the demonstration of which lies so deep as not to be discovered until after many fruitless efforts; and even then it is obtained by some tedious and artificial process, while the simpler methods of proof long remain hidden from us."† Or, again, let the young mathematician, who feels an instinctive liking for arithmetical enquiry, be encouraged by the observation which has been put on record by Jacobi in his brief notice of the life of Göpel, that many of those who have a natural turn for mathematical speculation find themselves in the first instance attracted by the Theory of Numbers.‡

There are three great departments of arithmetic, not, it must be admitted, wholly separable from one another, which seem to me at the present time to offer a very inviting field to the researches of the mathematician. Of these I will name first the arithmetical theory of homogeneous forms, or quantics, as we in England have now learned to call them. It is worthy of remembrance that some of the most fruitful conceptions of modern algebra had their origin in arithmetic, and not in geometry or even in the theory of equations. The characteristic properties of an invariant, and of a contravariant, appear with distinctness for the first time in the Disquisitiones Arithmeticæ;§ and in that treatise also the attention of mathematicians was for the first time directed to the study of quantics of any order and of any number of indeterminates.‖ Again, Eisenstein in the course of his researches on the arithmetical theory of binary cubic forms was led to the discovery of the first covariant ever considered in analysis—the Hessian of the

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* "Gauss. Zum Gedächtniss." Von W. Sartorius v. Waltershausen. Leipzig, 1856.

† Preface to Eisenstein's "Mathematische Abhandlungen." Berlin 1849.

‡ Notiz über A. Göpel, Crelle's Journal, Vol. xxxv. p. 313.

§ Disq. Arith., Arts. 157, 267, 268, where binary and ternary quadratic forms are considered.

‖ "Sed manifesto hoc argumentum" [the theory of binary quadratic forms] "tanquam sectionem maxime particularem disquisitionis generalissimæ de functionibus algebraicis rationalibus integris homogeneis plurium indeterminatarum et plurium dimensionum considerare . . . possumus." "Sufficiat hunc campum vastissimum geometrarum attentioni commendavisse, in quo materiem ingentem vires suas exercendi, arithmeticamque sublimiorem egregiis incrementis augendi invenient." (Disq. Arith., Art. 266.)

cubic form.* But the progress of modern algebra and of modern geometry have far outstripped the progress of arithmetic, and one great problem which arithmeticians have before them at the present time is to endeavour to turn to account for their own science the great results which have been obtained in the sister sciences. How difficult this problem may prove is perhaps best attested by the little progress that has been made towards its complete solution. One or two instances may serve to illustrate the actual position of the enquiry. The algebraical problem of the automorphics of a quadratic form, containing any number of indeterminates, may be regarded as completely solved by a formula due to M. Hermite and Professor Cayley.† The arithmetical formula which gives the automorphics of a binary quadratic form has long formed a part of the elements of the theory of numbers; and the corresponding investigation for an indefinite ternary quadratic form may be now regarded as completed by the memoirs in which M. Paul Bachmann has followed up the earlier researches of M. Hermite.‡ Again, the problem of the equivalence of two positive or definite ternary quadratic forms was completely solved by Seeber; and the problem of the arithmetical automorphics of such forms, by Eisenstein.§ The corresponding but far more difficult problem of equivalence for indefinite

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* See "Crelle's Journal," Vol. xxvii. p. 89. It is remarkable that the cubic covariant does not explicitly appear in this paper (December, 1843) or in the note (p. 105) which immediately follows it. This omission is supplied in a subsequent note, dated March 3, 1844 (*ibid.* p. 319). The earliest papers of Boole, who approached the study of linear transformations from a geometrical point of view, belong to the years 1841 and 1843 (Cambridge Mathematical Journal, Vol. ii. p. 64 and Vol. iii. p. 1 and p. 106). In these papers (of which perhaps only the first should be cited here) covariants do not appear, but the first general theorem of invariance ever enunciated, the theorem of the invariance of the discriminant of any quartic, is distinctly stated and proved.

† M. Hermite, in "Crelle's Journal," Vol. xlvii. p. 309, appears to have considered forms of three indeterminates only; his solution was subsequently generalized by Professor Cayley (*ibid.* Vol. i. p. 288). See also a later memoir by Professor Cayley "On the Automorphic Linear Transformation of a Bipartite Quadric Function," in the "Philosophical Transactions" for 1858.

‡ The solution of the problem is made to depend on the solution in integral numbers of the indeterminate equation $p^2 + F(q_1, q_2, q_3) = 1$, where F is the contravariant of the given ternary form. No general method, however, of obtaining the solutions of this equation has as yet been given. (See M. Hermite in the memoir already cited, "Crelle," Vol. xlvii. p. 307 *sqq.*; M. Bachmann, "Borchardt," Vol. lxxi. p. 296, and Vol. lxxvi. p. 331, together with the note by M. Hermite completing his former solution, *ibid.* Vol. lxxviii. p. 325.)

§ See L. Seeber, "Untersuchungen ueber die Eigenschaften der positiven ternären quadratischen Formen," Freiburg, 1831; and, in connection with this work, the review of it by Gauss (in the Göttingen "Gelehrte Anzeige" for 1831; or in "Crelle," Vol. xx. p. 312; or in the collected edition of Gauss' Works, Vol. ii. p. 188), and the subsequent and simpler investigations of Dirichlet ("Crelle," Vol. xl. p. 209), and M. Hermite (*ibid.*, p. 173). The theory of the automorphics of positive ternary quadratic forms is given by Eisenstein in the Appendix to his "Table of Reduced Positive Ternary Quadratic Forms." ("Crelle," Vol. xli. p. 227.) He observes (see the note at p. 230) that Seeber, without actually solving the problem, had come extremely near to its solution.

ternary forms has received its first solution only in very recent times from M. Eduard Selling;* and perhaps it is not too much to hope that these profound researches may receive some further development from their distinguished author, and may be brought into closer relation with other parts of arithmetical and algebraical theory. So far, then, as binary and ternary quadratic forms are concerned, we have not much reason to complain of the slowness of the advances made by arithmetic. But if we pass to quadratic forms of four or more indeterminates, we shall find that the limits within which our arithmetical knowledge is confined are indeed restricted. The fundamental theorem of M. Hermite, that the number of non-equivalent classes of quadratic forms having integral coefficients and a given discriminant is finite, and the recent researches of M. Zolotareff and Korkine on the minima of positive quadratic forms, mark the extremest limit to which enquiry has been pressed in this direction.†

As a second and much simpler instance of the difficulties which remain for arithmetic after the work of algebra is done, let us consider the system of two binary quadratic forms. The first question that we naturally ask, is, what is the arithmetical meaning of the evanescence of their joint invariant? I gave myself an answer to this question some years ago in the following theorem, which for brevity I express in the proper technical language.

"If the joint invariant of two properly primitive forms vanishes, the determinant of either of them is represented primitively by the duplicate of the other."‡

This theorem is very far from exhausting the subject to which it refers. But it may serve as a fair illustration of the class of enquiries which I wish to propose to the attention of the Society as likely to be not unfruitful. The geometrical interpretation of the invariantive character to which the theorem relates is (as we all know) that the two pairs of elements, represented by the two quantics, are harmonically conjugate; and I think it especially deserving of notice that the same invariantive character has an important meaning both in arithmetic and in geometry, but that neither of the two interpretations seems in the least likely to suggest the other. If we pass on to the case of two ternary quadratic forms, the geometrical signification of the evanescence of either of their joint invariants is now embodied in well-

* "Borchardt's Journal," Vol. lxxvii. p. 143.

† See the Letters of M. Hermite to Jacobi ("Crelle," Vol. xl. p. 261, *sqq.*), and the papers of M. Zolotareff and Korkine ("Clebsch," Vol. v. p. 581, and Vol. vi. p. 366). I may perhaps also be allowed to refer to my own papers "On the Orders and Genera of Quadratic Forms containing more than three Indeterminates," in the Proceedings of the Royal Society, Vol. xiii. p. 199, and Vol. xv. p. 387.

‡ Report on the Theory of Numbers in the Reports of the British Association for 1863, p. 783, Art. 123.

known elementary theorems; but I do not think that any answer has been given to the corresponding arithmetical question, nor indeed do I know that anyone has occupied himself with it. I would, however, venture to hazard a conjecture that the arithmetical interpretation of these invariantive conditions may have an important bearing on the researches of M. Selling, to which I have already referred.

I do not wish to weary the Society with too many particular examples; but I will venture to point to one more instance from which it would appear that modern geometrical and analytical concepceptions may help us a little, if only a little, on our way in the trackless wilds of arithmetic. Let us take a question which has some relation to the familiar notions of "unicursality" and "one-to-one correspondence." It is an old theorem, that if the homogeneous indeterminate equation of the second degree containing three variables admits of one solution, it admits of an infinite number; and there is a Memoir of Cauchy* showing how from one given solution all the solutions are to be derived. But here two things deserve our notice—(1) That no geometry (so far as I am aware) helps us in any way to decide whether the given equation does or does not admit of solution. The criterion turns on the definition (first given by Eisenstein) of the generic characters of ternary quadratic forms—a definition which itself depends on a simple arithmetical inference from the algebra of such forms.† But (2), in strong contrast to what I have just stated, when once we have a single solution, the rest is a matter of intuitive geometry. Our equation represents a conic; the conic is real, because we have a single rational point on it; every rational line drawn through this point meets the conic again in a rational point; and thus the unicursality of the conic enables us to deduce from any one solution all the solutions which exist, and, what is more, to obtain them all in a natural sequence. To pass to a question of a somewhat higher order of difficulty, there is no known criterion (so far as I am aware) for deciding whether a ternary homogeneous cubic equation does or does not admit of solution in integral numbers. Such a criterion would be of great interest, and ought not, one would suppose, to lie beyond the present scope of analysis.‡ But here again geometry shows us at once, that if we have one solution, we have, in general, an infinite number. For the tangential of a rational point on a rational cubic curve is itself a

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* "Exercices de Mathématiques," vol. i. p. 233. Cauchy considers the ternary cubic as well as the ternary quadratic equation in this memoir.

† See Eisenstein, in "Crelle," Vol. xxxv. p. 117; and a note of my own in the Proceedings of the Royal Society for 1864, Vol. xiii. p. 110.

‡ Special cases of the equation here considered have attracted much attention. It will be sufficient to mention Fermat's theorem of the impossibility of the equation $x^3 + y^3 + z^3 = 0$.

rational point, and the line joining two given rational points meets the cubic in a third rational point. (There are, of course, cases of exception to this mode of derivation of one integral solution from another, but I need not advert to them here.) Advancing a little further, we have not to look very far into the connexion which modern algebra has established between ternary cubic and binary quadratic forms in order to satisfy ourselves that the Diophantine problem of rendering a biquadratic expression a perfect square (a problem which has been the subject of numerous researches ever since the time of Euler) is the same as the problem of finding rational points upon a cubic curve of which the equation is rational; and that, in particular, the *tangential* method to which I have just referred enables us in general to deduce an infinite number of solutions of the Diophantine problem from any given solution.

A second department of Arithmetic which, as it seems to me, has in quite recent times received less attention than it deserves, is the Theory of congruences. Some time before the notation of congruences had been introduced into the Theory of Numbers by Gauss, Lagrange, to whom the conception of a congruence (apart from any special notation) was perfectly familiar, had established the elementary theorem: *

"If an expression of the form $\frac{f(x)}{\phi(x)}$ [where $f(x)$ and $\phi(x)$ are rational functions of x having integral coefficients] acquires an integral value for any given integral value of x, the value of $\phi(x)$ must be a divisor of the resultant of $f(x)$ and $\phi(x)$."

This theorem naturally suggests another which was subsequently given by Cauchy: †

"If two congruences which have the same modulus admit of a common solution, the modulus is a divisor of their resultant."

These propositions suggest the possibility of transferring to the Theory of Numbers some at least of the results which have been obtained by modern researches in the theory of algebraical elimination. For example, we are led to consider the problem:

"Given two congruences, to find the number of their common roots when we take in succession for modulus each divisor of their resultant."

In all such inquiries we shall find that the considerations which suffice for the solution of the algebraical problem enter as indispensable elements into the arithmetical investigation, but that this investigation

* The "Disquisitiones Arithmeticæ" were published in 1801. The Memoir of Lagrange, entitled "Nouvelle méthode pour résoudre les Problèmes indéterminés," appeared in the Transactions of the Academy of Berlin for 1768. See also paragraph 4 (p. 528) of his Additions to the French translation of the Algebra of Euler (Lyons, 1774, and often reprinted since); the reference here is to the edition of the "an iii de l'ère républicaine."

† "Exercices de Mathématiques," vol. i. p. 164.

compels us to take notice of other elements also, with which, in algebra, we were not concerned. Thus, in the solution of the problem which I have mentioned, and which I hope at some future day to bring more fully under the notice of the Society, we should have not only to consider in a general manner the system of the divisors of the resultant itself, but we should also have to distinguish, in that system of numbers, those which are at the same time common divisors of certain systems of minors in the dialytic matrix of which the resultant is the determinant.

If I may be allowed to regard the subject of complex numbers as belonging to the theory of congruences, I must also be allowed to modify to a certain extent what I have said as to the indifference with which that theory has been looked upon in very recent times; for there is no reason to complain that complex numbers have received insufficient attention, at least from the mathematicians of the Continent. Without referring here to the results obtained by Lejeune Dirichlet, or to the splendid series of researches upon the complex numbers formed with roots of unity which we owe to M. Kummer, we may notice that the general theory has attracted the attention of M. Kronecker, whose investigations relating to it have unfortunately not as yet been published, and are only known from the application which he has made of them to the equations which present themselves in the theory of elliptic functions.* In a Supplement, added to the second edition of Lejeune Dirichlet's Lectures on the Theory of Numbers, M. Dedekind has given the outlines of a complete and very original theory of complex numbers, in which he has to a certain extent deviated from the course pursued by M. Kummer, and has avoided the introduction (at least in a formal manner) of ideal numbers.† I may remind my hearers that the inapplicability, in general, of Euclid's theory of the greatest common divisor to complex numbers formed with the roots of equations having integral coefficients renders it impossible to define the prime factors of such numbers in the same way in which we can define the prime factors of common integers, or of complex numbers of the form $a+b\sqrt{-1}$; and that, in the effort to overcome the difficulty thus arising, M. Kummer was led to introduce into arithmetic an entirely new and very important conception—that of ideal numbers. I shall ask leave to mention a very recent and very interesting application of this conception which has been made by M. Zolotareff to the solution

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* See the "Monatsberichte" of the Academy of Berlin for June 26, 1862, p. 370. M. Dedekind also refers to the investigations of M. Kronecker at p. viii. of his Preface to the second edition of Dirichlet's Lectures, presently to be noticed.

† [M. Dedekind has also commenced the publication of a *résumé* of his theory in the "Bulletin des Sciences Mathématiques et Astronomiques" for December, 1876.]

of a problem of the Integral Calculus, which was first attempted by Abel, and has since attracted the attention of MM. Tchébychef and Weierstrass. We owe to Abel the remarkable theorem that the differential expression

$$du = \frac{(x+\lambda)\,dx}{\sqrt{(x^4+ax^3+bx^2+cx+d)}}$$

can or cannot be integrated by logarithms, for some value of the parameter λ, according as the radical can or cannot be developed in a periodic continued fraction. But Abel gave no criterion for deciding whether the development is or is not periodic—a question which obviously cannot be decided by mere trial. The problem was first solved by M. Tchébychef for the case in which the coefficients a, b, c, d are rational; and the complete solution has at last been obtained by M. Zolotareff, with the help of a new theory of complex ideal numbers.*

The last part of arithmetical theory to which I would wish to direct the attention of some of the younger mathematicians of this country is the determination of the mean or asymptotic values of arithmetical functions. This is a field of inquiry which presents great difficulties of its own; and it is certainly one in which the investigator will not find himself incommoded by the number of his fellow-workers. "Vix

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* The Memoir of Abel, "Sur l'intégration de la formule différentielle $\frac{\rho\,dx}{\sqrt{R}}$, R et ρ étant des fonctions entières," will be found in "Crelle," vol. i., p. 185, or "Œuvres Complètes," vol. i. p. 33. In this memoir Abel demonstrates the general theorem that, if R be a rational and integral function of x of any order, such that $\sqrt{R}$ can be expressed by a symmetrical and periodic continued fraction of the form

$$(1)\ldots\ldots\ldots \sqrt{R} = r + \frac{1}{2\mu+}\ \frac{1}{2\mu_1+}\ \ldots\ldots\ \frac{1}{2\mu_1+}\ \frac{1}{2\mu+}\ \frac{1}{2r+}\ \ldots\ldots,$$

it is always possible to find a rational and integral function ρ of x such that

$$(2)\ldots\ldots\ldots \int \frac{\rho\,dx}{\sqrt{R}} = \log\left(\frac{y+\sqrt{R}}{y-\sqrt{R}}\right) + C,$$

y being a rational function of x; and that, conversely, if the equation (2) can be satisfied, the development of $\sqrt{R}$ is of the form (1). Abel further shows that y is always the convergent $[r, 2\mu, 2\mu_1, \ldots 2\mu_1, 2\mu]$; and he gives the application of his theorem to the case in which R is a biquadratic function. The memoir of M. Tchébychef was first published in the "Bulletin de l'Académie de St. Pétersbourg" for 1860 (tom. iii. pp. 1—12), and has been reprinted in the "Mélanges Mathématiques et Astronomiques de St. Pétersbourg" (tom. iii. pp. 182—192), and also in "Liouville's Journal" (Second Series, 1864), vol. ix. p. 225. M. Tchébychef has given his method without demonstration; this was first supplied by M. Zolotareff in a paper, "Sur la méthode d'intégration de M. Tchébychef" ("Mathematische Annalen," vol. v. p. 560). The complete solution of the problem is contained in a work by M. Zolotareff ["Théorie des Nombres entiers complexes," St. Pétersbourg,] which I have not yet seen. The result obtained by M. Weierstrass (however remarkable in itself), that the integrability by logarithms of the differential du depends on the possibility of expressing a certain constant in the form $\alpha K + i\beta K'$, where α and β are rational numbers, and K and K′ are certain complete elliptic integrals, does not supply an available criterion. (See the "Fortschritte der Mathematik," vol. vi. p. 118.)

ullus reperietur geometra," wrote Euler* in the last century, "qui non, ordinem numerorum primorum investigando, haud parum temporis inutiliter consumpserit." But I do not think that, as a rule, the mathematicians of the present day have any reason to reproach themselves on this score, or stand in any need of the apology which Euler proceeds to deliver. Nevertheless, much has been done in this direction since the days of Euler; enough, certainly, to give abundant encouragement to further inquiry. The first asymptotic results that were obtained are due to Gauss, and are given without demonstration in the "Disquisitiones Arithmeticæ";† they relate to the average number of classes of binary quadratic forms of a positive or negative determinant. The general principles on which such inquiries depend were laid down by Lejeune Dirichlet, forty-eight years later, in a memoir entitled "Ueber die Bestimmung der mittleren Werthe in der Zahlentheorie," and inserted in the Transactions of the Berlin Academy for 1849. The subject has recently been resumed by M. Mertens of Cracow, who, in an interesting memoir (Borchardt's Journal, vol. lxxvii., p. 289), has determined the asymptotic values of several numerical functions. In particular, he has demonstrated the expression given by Gauss for the mean value of the number of quadratic forms of a negative determinant; this mean value being, in the vicinity of n, when n is a considerable number, the quotient obtained by dividing $\frac{2}{\pi}\sqrt{n}$ by the sum of the cubes of the reciprocals of the natural numbers.

As to our knowledge of the series of the prime numbers themselves, the advance since the time of Euler has been great, if we think of the difficulty of the problem; but very small if we compare what has been done with what still remains to do. We may mention, in the first place, the undemonstrated, and indeed conjectural, theorems of Gauss and Legendre as to the asymptotic value of the number of primes inferior to a given limit x (the former theorem assigns for this value the integral logarithm $\mathrm{Li}(x) = \int_0^x \frac{dx}{\log x}$, the latter an expression of the form $\frac{x}{\log x - a}$).‡ But these theorems are only approximately

* Dec. 1, 1760, "Novi Commentarii Petropolitani," vol. ix. p. 99; or, "Commentationes Arithmeticæ Collectæ" (Petropoli, 1849), vol. i. p. 356.

† See Art. 302, 304, and the Additamenta to Art. 306, X; also various passages in the posthumous fragments, "De nexu inter multitudinem classium," &c. (Gauss, "Werke," vol. ii. pp. 269—303.)

‡ See Legendre, "Théorie des Nombres" (Paris, 1830), vol. ii. p. 65; Gauss to Encke (Dec. 24, 1849), in Gauss' Collected Works, vol. ii. p. 444. Legendre assigns to a the conjectural value 1·08366. Gauss compares his own formula and that of Legendre with the results of actual enumerations of the prime numbers, and finds that, as far as the end of the third million, the comparison is in favour of the formula of Legendre, but that the error of that formula shows a tendency to increase

consistent with one another, and are perhaps still less approximately true. The memoir of Bernhard Riemann, "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse," contains (so far as I am aware) the only investigation of the asymptotic frequency of the primes which can be regarded as rigorous.* He shows that, if $F(x)$ be the number of primes inferior to x, there exists an analytical expression for the series

$$F(x) + \tfrac{1}{2}F(x^{\frac{1}{2}}) + \tfrac{1}{3}F(x^{\frac{1}{3}}) + \ldots\ldots,$$

which consists (1) of a term which does not increase without limit with x; (2) of a non-periodic term $Li(x)$; (3) of an infinite series of periodic terms of the type $Li(x^{\frac{1}{2}-\alpha i}) - Li(x^{\frac{1}{2}+\alpha i})$, the constants α being the roots (infinite in number) of a certain transcendental equation. It follows that the non-periodic part of the expression for $F(x)$ is of the type

$$Li(x) - \tfrac{1}{2}Li(x^{\frac{1}{2}}) - \tfrac{1}{3}Li(x^{\frac{1}{3}}) + \tfrac{1}{6}Li(x^{\frac{1}{6}}) - \tfrac{1}{7}Li(x^{\frac{1}{7}}) + \ldots\ldots;$$

and thus the equation of Gauss, $F(x) = Li(x)$, can, roughly speaking, be correct only as far as quantities of the order $x^{\frac{1}{2}}$.

No less important than the investigation of Riemann, but approaching the problem of the asymptotic law of the series of primes from a different side, is the celebrated memoir, "Sur les Nombres Premiers," by M. Tchébychef,† in which he has established the existence of limits within which the sum of the logarithms of the primes P inferior to a given number x must be comprised. The limits assigned by M. Tchébychef are not very close, not even close enough to determine the asymptotic value of the quotient $\frac{\Sigma \log P}{x}$, to which the value 1 has been conjecturally assigned. But the theorem of M. Tchébychef may be, perhaps, said to mark the furthest point to which our knowledge of the series of prime numbers has yet been carried; and while it is truly

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more rapidly than the error of his own. He also observes that the mean value of the constant a tends to decrease, and that its true limit may possibly be 1, or may differ from 1 by a quantity of the order $\frac{1}{\log x}$. Encke had communicated to Gauss a formula of his own, $\frac{x}{\log x} \times 10^{\frac{1}{2 \log x}}$, which, as Gauss observes, may, for very great values of x, be regarded as coinciding with $\frac{x}{\log x - \frac{1}{2}\log 10}$. (See the letter cited, and also the note of M. Schering, *ibid.* p. 521.)

* See the "Monatsberichte" of the Academy of Berlin for November, 1859; or Riemann's "Mathematische Werke," p. 137. In the "Annali di Matematica," tom. iii. p. 52 (1860), M. Genocchi has given a very interesting account of the method of Riemann, and has arrived at a result differing in one respect (see p. 58) from that of Riemann.

† "Liouville," First Series, vol. xvii. p. 366. The memoir was presented to the Academy of St. Petersburg in 1850.

remarkable that, in a matter of so much difficulty, a process so apparently simple as that which he has employed should be capable of leading to a result of so much interest and importance, it is somewhat disappointing to find that this method, even in the hands of its eminent inventor, should seem incapable of being pursued further, and unlikely to furnish any nearer approximation to the truth. The method of M. Tchébychef, profound and inimitable as it is, is in point of fact of a very elementary character, and in this respect contrasts strongly with that of Riemann, which depends throughout on very abstruse theorems of the Integral Calculus.

I do not know that the great achievements of such men as Tchébychef and Riemann can fairly be cited to encourage less highly gifted investigators; but at least they may serve to show two things—first, that nature has placed no insuperable barrier against the further advance of mathematical science in this direction; and, secondly, that the boundaries of our present knowledge lie so close at hand that the inquirer has no very long journey to take before he finds himself in the unknown land. It is this peculiarity, perhaps, which gives such perpetual freshness to the higher arithmetic. It is one of the oldest branches, perhaps the very oldest branch, of human knowledge; and yet some of its most abstruse secrets lie close to its tritest truths. I do not know that a more striking example of this could be found than that which is furnished by the theorem of M. Tchébychef. To understand his demonstration requires only such algebra and arithmetic as are at the command of many a schoolboy; and the method itself might have been invented by a schoolboy, if there were again a schoolboy with such an early maturity of genius as characterised Pascal, Gauss, or Evariste Galois.*

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* In addition to the memoirs to which reference has been made in the text, we may also mention the following:—(1) M. Tchébychef, "Sur la totalité des nombres premiers inférieurs à une limite donnée" ("Liouville," 1st Series, vol. xvii. p. 341). In this paper (which was presented to the Academy of St. Petersburg in 1848) M. Tchebychef proves (among other things) that, if the expression $\frac{x}{\mathrm{F}(x)} - \log x$ has a limit at all, the value of that limit must be -1. This result shows that in the approximate formula of Legendre, $\mathrm{F}(x) = \frac{x}{\log x - a}$, we ought to take $a = +1$. (2) A paper by the late Judge Hargreave, in the "Philosophical Magazine" for 1849 (vol. xxxv. p. 36), in which it is shown (but not by any very rigorous demonstration) that the average interval between two consecutive primes in the vicinity of any very great number x is $\log x$; a result at which Gauss had arrived while still a boy, as may be inferred from his letter to Encke, quoted above. (3) A paper in the "Mathematische Annalen," vol. ii. p. 636, in which the author, M. Meissel, by the aid of a method suggested by those employed by Legendre ("Théorie des Nombres," vol. ii. p. 86), obtains a formula which greatly facilitates the determination of the number of prime numbers contained between given limits; he has thus found that the number of primes in the first ten millions is 664579; and in a later note ("Mathematische Annalen," vol. iii. p. 523) that the number of primes in the

I pass on to speak of some other branches of analysis which appear to me at the present moment to promise much in the immediate future.

I will first refer to one or two points to which the transition from the arithmetic of whole numbers is easy and natural.

We owe to Jacobi the first suggestion of a method of approximation which forms a natural extension of the theory of continued fractions, but which still remains in an incomplete condition. In the memoir "De functionibus duorum variabilium quadrupliciter periodicis," ("Crelle," Vol. xiii. p. 55,) Jacobi demonstrated the theorem that an uniform function of a single variable can at most be doubly periodic; and that, if it be doubly periodic, the ratio of the two periods is necessarily imaginary. He effected this by proving that, if $aa'a''$, $bb'b''$ are independent irrational quantities, it is always possible to find integral numbers $mm'm''$ such that the value of each of the two expressions

$$ma+m'a'+m''a'',$$
$$mb+m'b'+m''b'',$$

shall be less than any quantity that can be assigned.

This idea of Jacobi was subsequently further developed by M. Hermite, who showed its connexion with the theory of the reduction of quadratic forms (see his letters to Jacobi in "Crelle's Journal," Vol. xl. p. 261 *sqq.*) The same conception lies at the basis of Lejeune Dirichlet's researches on complex units, and led him to his celebrated generalization of the theory of the Pellian Equation.*

Since the death of Jacobi, a memoir of his (apparently left incomplete) has been published in "Borchardt's Journal" (Vol. lxix. p. 29), in which he examines the relations between the successive sets of integral numbers x_0, x_1, x_2, which render an expression such as

$$x_0+x_1\omega_1+x_2\omega_2$$

(where ω_1 and ω_2 are irrational quantities) approximately equal to zero. He applies the theory to the examples $\omega_1 = 2^{\frac{1}{3}}, = 3^{\frac{1}{3}}, = 5^{\frac{1}{3}}$; $\omega_2 = 2^{\frac{2}{3}}, = 3^{\frac{2}{3}}, = 5^{\frac{2}{3}}$, and finds that in each case the development is

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first hundred millions is 5,761,460. (4) A note by Mr. J. W. L. Glaisher in the Report of the British Association for 1872 ("Transactions of the Sections," p. 19), in which the results of some enumerations of the primes are given, and are compared with Mr. Hargreave's theorem as to their average frequency. (5) A paper by M. Mertens ("Borchardt's Journal," vol. lxxviii. p. 46), in which he determines the asymptotic values of the functions $\Sigma\frac{1}{P}$, $\Pi\left(1-\frac{1}{P}\right)$. These functions had been already considered by Legendre ("Théorie des Nombres," vol. ii. p. 67), and by M. Tchébychef in the memoir already cited in this note; but M. Mertens obtains a more precise result by more rigorous reasoning. [(6) A preliminary note on an enumeration of primes, by Mr. Glaisher, in the Proceedings of the Cambridge Philosophical Society, Dec. 4, 1876.]

* See the "Monatsberichte" of the Academy of Berlin for 1842, p. 95, and for 1846, p. 105.

periodic; but he appears not to have obtained any demonstration of the general theorem that the corresponding development in the case of the root of any cubic equation having integral coefficients is always periodic.* These unfinished researches of Jacobi, to which M. Borchardt has called the special attention of mathematicians (in the preface to the 68th volume of his Journal) have been resumed by M. Bachmann,† and still more recently, though from a somewhat different point of view, by M. Fürstenau. The latter of these mathematicians defines a continued fraction of the second order to be a continued fraction in which each element is itself a continued fraction; and, availing himself of this definition, he has succeeded in showing that we can always approximate to the real root of an equation of the order n, having integral coefficients, by means of a periodic continued fraction of the order $n-1$.‡ It is evident that the discovery of such a mode of approximation to the root of an equation may lead to theoretical considerations of great interest, though it is hardly likely that the method itself will be found practically useful.

Closely allied to the investigation of new methods of approximation is the problem of determining the arithmetical or transcendental character of irrational quantities. It was first shown by M. Liouville § that irrational quantities exist which cannot be the roots of any equation whatever having integral coefficients; a proposition which certainly required the proof which it has received from him, although it might easily seem incredible *à priori* that such irrational quantities should not exist. We may, perhaps, be allowed to designate by the terms arithmetical and transcendental the two classes of irrational quantities between which the theorem of M. Liouville has taught us to distinguish; and it becomes a problem of great interest to decide to which of these two classes we are to assign the irrational numbers, such as ϵ and π, which have acquired a fundamental importance in analysis. To Lambert, the eminent Berlin mathematician of the last century, the first great step in this direction is due. He showed that neither π nor π^2 is rational; with regard to ϵ he was even more successful, for he was able to prove that no power of ϵ, of which the exponent is rational, can itself be rational.‖ There (with one slight exception) the question remained for more than a century; and it was reserved for M. Hermite,

* This theorem had been already obtained by M. Hermite. See the Letters already cited, "Crelle," Vol. xl. pp. 286—289.

† "Borchardt's Journal," Vol. lxxv. p. 25.

‡ E. Fürstenau, "Ueber Kettenbrücke höherer Ordnung," Wiesbaden. I regret to say that I only know this work from the notice in the "Jahrbuch über die Fortschritte der Mathematik" for 1874.

§ "Comptes Rendus," Vol. xviii. (1844), p. 883, and p. 910; reproduced with additions in "Liouville's Journal" (1st series), Vol. xvi. p. 133.

‖ "Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques," in the "Mémoires de l'Académie des Sciences de Berlin"

in the year 1873, to complete by a singularly profound and beautiful analysis, the exponential theorem of Lambert, and to prove that the base of the Napierian logarithms is a transcendental irrational.* But, in the letter to M. Borchardt already cited, M. Hermite declines to enter on a similar research with regard to the number π. "Je ne me hasarderai point," he says, "à la recherche d'une démonstration de la transcendance du nombre π. Que d'autres tentent l'entreprise; nul ne sera plus heureux que moi de leur succès; mais croyez m'en, mon cher ami, il ne laissera pas que de leur en coûter quelques efforts." It is a little mortifying to the pride which mathematicians naturally feel in the advance of their science to find that no progress should have been made for one hundred years and more toward answering the last question which still remains to be answered with regard to the quadrature and rectification of the circle. But mathematical discovery is like electricity; it follows the lines of least resistance; and an adherence to the rule which this analogy suggests is certainly conducive to the comfort of the individual mathematician, and is probably also, in the long run, conducive to the progress of mathematics themselves. It has often happened in mathematical history that a difficulty, which had for ages

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for 1761, p. 265; the demonstration depends on the continued fraction

$$\frac{\epsilon^x - \epsilon^{-x}}{\epsilon^x + \epsilon^{-x}} = \frac{x}{1+}\ \frac{x^2}{3+}\ \frac{x^2}{5+}\ \frac{x^2}{7+\ldots}.$$

A different method of proving the incommensurability of ϵ (depending on the exponential series) has found its way into many elementary treatises; it would seem that this simple method cannot be applied to prove the more general proposition that $\epsilon^{\frac{m}{n}}$ is incommensurable; but it has been successfully employed by M. Liouville ("Liouville's Journal," 1st series, Vol. v. p. 192) to show that neither ϵ nor ϵ^2 can be the root of a rational quadratic equation. This result forms the only extension which the theorem of Lambert had received up to the date of the memoir of M. Hermite, "Sur la fonction exponentielle," to which we shall presently refer. The only elementary work in which (so far as I know) the incommensurability of $\epsilon^{\frac{m}{n}}$ is demonstrated, is Mr. Todhunter's "Algebra," ed. 5, p. 530. The theorems as to the incommensurability of π and π^2 are excluded from English text-books; the only exception that occurs to me being Sir David Brewster's English edition (Edinburgh, 1824) of the Geometry of Legendre, where Legendre's reproduction of the demonstration of Lambert is given in Note iv. p. 239. The exclusion of these theorems is a matter of regret; for they constitute the only "short method with the circle-squarers"; and perhaps the extraordinary prevalence within the United Kingdom of the form of delusion known as circle-squaring may partly arise from the appearance of an "ipsi dixerunt" on the part of the mathematicians, which is certainly suggested by the omission in elementary works of any rigorous demonstration of the irrationality of π. M. Hermite has given a demonstration of the irrationality of π and π^2, which is very beautiful and entirely different from that of Lambert (Letter to M. Borchardt in "Borchardt's Journal," Vol. lxxvi. p. 342); with this single exception, the theory of the quadrature of the circle rests to-day where Lambert left it in the year 1761.

* See the Memoir "Sur la fonction exponentielle," already cited in the preceding note, "Comptes Rendus," Vol. lxxvii., pp. 18 etc.; and also published separately by Gauthier-Villars, 1874.

resisted all direct attempts to overcome it, has yielded at last to the gradual advance of science; just as in the operations of strategy a strong position, which cannot be carried by a front attack, may nevertheless be turned and taken in the rear by an enemy who has possessed himself of the country round it.*

I may, perhaps, mention yet one more class of questions lying on the border land of arithmetic and algebraic analysis; I mean the questions which relate to the transcendental or algebraic character of developments in the form of infinite series, infinite products, or infinite continued fractions. The theorems of Eisenstein and M. Heine, of which a simple and beautiful demonstration has lately been laid before us by our colleague M. Hermite, are amply sufficient to awaken the expectation of great future discoveries in this almost unexplored field of enquiry.†

I have detained you so long over arithmetical and quasi-arithmetical subjects that I can only venture to glance hastily at some topics on which I could have wished to have dwelt much longer. I am afraid that I have only given you an additional instance of that one-sidedness against which, as I have reminded you, we were cautioned by our first President. In the hope of convincing you that I have not wholly forgotten the claims of other parts of our science, I will now hazard the assertion, that (after all) the advancement of the Integral Calculus is at once the most arduous and the most important task to which a mathematician can address himself. In the applications of mathematics to physics the integral calculus is confessedly of ever increasing importance; and it is especially interesting to observe that some of the most recent developments which it has received have had their origin in considerations of pure analysis, and yet have come just in time to furnish us with the most appropriate instruments for dealing with the problems which at the present moment are the most prominent in physical enquiries. But I must not dwell on the prospects of great future extension which are thus opened up for the various branches of mathematical physics; I can only advert (and that very hastily) to some of those parts of the integral calculus which, even from the point of view of the pure mathematician, seem to promise an abundant and immediate harvest.

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* I find that I am here closely following (haud passibus æquis) some observations of the late Dr. Hermann Hankel in his inaugural address, "Die Entwickelung der Mathematik in der letzten Jahrhunderten," Tübingen, 1869. This discourse, by one who was at once a learned scholar and an original investigator, contains much which deserves the attention of all who are interested in the progress of mathematical science, or who wish to see a higher spirit infused into the mathematical teaching given in the schools and universities of this country.

† Eisenstein, "Monatsberichte" of the Berlin Academy for 1852, p. 441; M. Heine in "Crelle's Journal," Vol. xlv. p. 285, and Vol. xlviii. p. 267. The note of Eisenstein is reproduced in the first of these memoirs.

Let me first mention the theory of ordinary differential equations. This is a subject which ought to have a special interest for ourselves, as one of the latest advances that have been made in it—the introduction of symbolical methods- is due in great measure to English mathematicians, and above all others to George Boole. Nor have Englishmen been behindhand in the cultivation of another branch of the subject (intimately connected with the use of symbolical methods) —the representation of the solutions of differential equations by means of definite integrals. But, simultaneously with these investigations, a line of research has been pursued on the Continent to which we in England have not paid equal attention. I refer to the endeavours which have been made to determine the nature of the function defined by a differential equation, from the differential equation itself, and not from any analytical expression of the function, obtained by first "solving" the differential equation. The generality and importance of such an inquiry (whatever be its difficulty) cannot be overrated; for, just as we long since learned to regard integration in a finite form (or, more properly, integration by means of algebraic or exponential and logarithmic functions) as only a very small part of the problem which the Integral Calculus has to solve with regard to differential expressions containing a single variable, so also, when we come to differential equations, we are forced to remember that the variety and complexity of the functional relations expressed by them may altogether transcend any other means of expression at our disposal. Perhaps we may regard as the fundamental theorem in the whole subject the proposition of Cauchy, that every differential equation admits (in the vicinity of any non-singular point) of an integral which is synectic within a certain circle of convergence, and which is consequently (within that circle) developable by the series of Taylor. Various applications of this theorem (together with a demonstration somewhat simpler than that given by Cauchy) will be found in the classical treatise of MM. Briot and Bouquet.* Closely allied to the point of view indicated by the theorem of Cauchy is that adopted by Riemann, who regards a function of a single variable as defined by the position and nature of its singularities, and who has applied this conception to the linear differential equation of the second order which is satisfied by the hypergeometrical series. In the memoir, "Beiträge zur Theorie der durch die Gauss'sche Reihe $F(\alpha, \beta, \gamma, x)$ darstellbaren Functionen,"† Rie-

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* "Théorie des Fonctions Elliptiques," ed. 2, Paris, 1875, p. 325. See also a memoir by the same authors in the "Journal de l'Ecole Polytechnique," cahier 36, p. 137. For an enumeration of Cauchy's Memoirs on Differential Equations, see his Life by M. Valson, Paris, 1868, vol. ii., cap. 9, pp. 104—117.

† "Transactions of the Academy of Göttingen" for 1875, vol. vii., or in Riemann's Collected Works (Leipzig, 1876).

mann sets out with the conception of a function which possesses three discriminantal points (I venture to propose this word as the most natural English rendering of "Verzweigungs-punkte"), and which is further characterised by the property that any three of the values which it admits at any point are connected by a linear and homogeneous equation with constant coefficients. Such a function Riemann shows, by reasoning of great beauty and originality, necessarily satisfies the linear differential equation of the hypergeometric series; and thus the nature and mode of existence of the functions defined by that equation are put before us with a precision and clearness which could not, perhaps, have been attained by any application of the ordinary methods of analysis to the discussion or integration of the equation.

The collected works of Riemann include another, but unfortunately unfinished memoir ["Zwei allgemeine Lehrsätze ueber lineäre Differential-gleichungen mit algebraischen Coefficienten"], relating to the case in which the number of independent functional values is any whatever instead of only three. And the fertility of the conceptions of Cauchy and of Riemann is further attested by the researches to which they have given rise, and are still giving rise, in Germany—researches among which I must especially mention those of L. Fuchs, whose papers on linear differential equations, in the 66th and subsequent volumes of "Borchardt's Journal," must form, it seems to me, the basis of all future inquiries on this part of the subject.

There is one celebrated problem connected with differential equations which, after all that has been written and said about it, remains a problem still; I mean the problem of Singular Solutions. If it were not for the papers of M. Darboux ("Bulletin des Sciences Mathématiques et Astronomiques," Tom. iv., p. 158), and of Professor Cayley ("Messenger of Mathematics," Vol. ii. p. 6., and Vol. vi. p. 23), I do not know where I should advise a student to turn to acquire any distinctness of insight into this important question. These papers have at any rate rendered one great service; they clearly show that there is a difficulty, and a difficulty not yet surmounted. The point of the difficulty I presume to be, that whereas a singular solution, from the point of view of the integrated equation, ought to be a phenomenon of universal, or at least of general occurrence, it is, on the other hand, a very special and exceptional phenomenon from the point of view of the differential equation. The explanation suggested by M. Darboux is (to say the least) deserving of very careful consideration. He says, at p. 167 of the memoir just cited, "Since differential equations are formed by the elimination of constants from an equation in finite terms and its derived equations, writers have supposed (and it would seem erroneously) that, when we are given a differential equation of the first order (for example), it always possesses

an integral of the first order expressible in the form $f(xyc) = 0$, where f is a function having in the whole extent of the plane the properties generally recognised in analytical functions. This function f might be more or less difficult to find, but it was conceived of in every case as existing. Now this is just the disputable point, and we think that recent researches on the theory of functions ought to lead us to adopt a different view." It is evident that, if the observations of M. Darboux are well founded, an important series of questions arises as to the nature of the integral equation answering to a given differential equation; and, further, that some of the elementary considerations with which it is usual to introduce the subject of differential equations must be abandoned as untenable. The rules that can be given in aid of mathematical discovery are, I suppose, very few, and I have already ventured to call your attention to one of them—the rule that bids us follow up any opening that may present itself, rather than try to force a way against obstacles which may prove insurmountable. In the case before us, I think we come upon an illustration of another rule, which is of less general application, but nevertheless often useful. The rule is, that an apparent contradiction (as distinct from a mere misunderstanding) is always to be regarded as an indication of some undiscovered truth. Yet it is remarkable what a tendency there is in the minds of men to ignore or soften down such apparent contradictions, instead of looking for the reality which lies at the bottom of them. The equation $\int_1^x \frac{dx}{x} = \log x$ must have been familiar to mathematicians for a century at least before they set themselves seriously to examine the apparent contradiction presented by the equation of a single-valued to a multiple-valued expression. And yet what a flood of light was thrown on the whole theory of functions by the researches to which Cauchy and others were led when the endeavour was at last made to account for this and similar apparently inexplicable phenomena. If the clue offered by such familiar instances as the equations

$$\int_1^x \frac{dx}{x} = \log x, \quad \int_0^x \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x,$$

had been seized and followed up, it is difficult to believe that the main outlines of the theory of elliptic functions would not have been discovered much sooner than they actually were. If we look back on the history of the past, the discovery of the "principle of double periodicity," and, with it, of the essential characteristic of an elliptic function, cannot but appear to us as one of the most extraordinary efforts of mathematical genius. The integration of the differential equation of elliptic integrals by Euler—an integration obtained by a sort of divination, which has deservedly remained celebrated in the history of

science; the systematization of the calculus of elliptic integrals by Legendre; the simultaneous discovery by Jacobi and Abel of the double periodicity latent in the equation of Euler,—these were the successive steps—and each one a gigantic step—by which those great mathematicians arrived at the theory of elliptic functions in the form in which we now possess it. But if we compare the actual history of the discovery with the outlines of the theory, as we find them, for example, in the work of MM. Briot and Bouquet, it is impossible not to be struck with the contrast. Each step in the theory, as exhibited in that work, appears to follow from those that precede it in such a natural and necessary order that we are inclined to wonder why those who discovered the great results themselves should have failed to find the easiest path of access to them.

If I had had the honour of addressing the Mathematical Society ten years ago, I think I should have had to complain of the neglect in England of the study of elliptic functions. But I cannot do so now. The University of Cambridge has given this subject a place in its Mathematical Tripos; the University of London in its examination for the Doctorate of Science. The British Association has supplied the funds requisite to defray the cost of printing Tables of the Theta function—Tables of which the mathematicians of this country may justly be proud, and which will form an enduring memorial of the great ability and indefatigable industry of our colleague, Mr. Glaisher. We further owe to Professor Cayley an introductory treatise on elliptic functions, the first which has appeared in our language. I consider that the service which he has thus rendered to students is an important one, and one for which we ought to be very grateful. I am convinced that nothing so hinders the progress of mathematical science in England as the want of advanced treatises on mathematical subjects. We yield the palm to no European nation for the number and excellence of our text-books of the second grade—I mean, such text-books as are intended to guide the studies of the undergraduate within the courses prescribed by our University examinations in honours. But we want works adapted to the requirements of the student when his examinations are over—works which will carry him to the frontiers of knowledge in various directions, which will direct him to the problems which he ought to select as the objects of his own researches, and which will free his mind from the narrow views he is too apt to contract while "getting up" subjects with a view to passing an examination, or, a little later in his life, while preparing others for examination. Can we doubt that much of the preference for geometrical and algebraical speculation which we notice among our younger mathematicians is due to the admirable works of Dr. Salmon; and can we also doubt that, if other parts of mathematical science had been equally fortunate in finding an

expositor, we should observe a wider interest in, and a juster appreciation of, the progress which has been achieved? There are, of course, other treatises besides those of Professor Cayley and Dr. Salmon to which I might refer; there is, for example, the work of Boole on Differential Equations; and there are the great historical treatises of Mr. Todhunter, so suggestive of research and so full of its spirit; we have also a recent work by the same author on the functions of Laplace, Lamé, and Bessel. But the field is not nearly covered, though, indeed, my enumeration is not complete; and, even without leaving the domain of the Integral Calculus, I might point out that there are at least three treatises which we greatly need—one on Definite Integrals, one on the Theory of Functions in the sense in which that phrase is understood by the school of Cauchy and of Riemann, and one (though he should be a bold man who would undertake the task) on the Hyperelliptic and Abelian Integrals. I fear that our colleague, Professor Clifford, would hardly listen to us if we were to appeal to him to undertake this task; but at least we may express the hope that he may be able to continue the profound researches which he has commenced on this great branch of analysis.

I feel I must now bring these somewhat desultory remarks to a conclusion; though, if your time and patience were unlimited, there are many things I could wish to say. Among other matters I should have adverted to the great efforts which have been made in very recent times, in Germany, in Russia, and in Norway, to advance the theory of Partial Differential equations; and I should have noted with pleasure that our own Society has received important communications on this subject from Professor Tanner. And again, leaving the field of the Integral Calculus, I think I should have hazarded some references to the Theory of Substitutions, and its applications to the Theory of Equations; and though I should have been relapsing into a region dangerously near to the Theory of Numbers, I should have exhorted the younger mathematicians of our time not to turn away from a subject which, if forbidding at the first aspect, contains so much promise of future development, and lies so near the very centre and fountain of much that is important in Algebra. Lastly, I should have endeavoured to make my peace with Geometry, which all this time I have been treating with such marked neglect, and would have invited your attention, though it were but for a few moments, to some of those questions of geometry which the natural advance of science has brought to the forefront at the present time. Even here, I might have found one example more of a study which we in England too much neglect; and I might perhaps have reminded you of the great hopes which Gauss entertained of the Geometry of Situation, containing, according to him, vast and as yet quite uncultivated regions over which our present analytical methods can

pretend to no dominion. I certainly should not have forgotten to congratulate the Society on the part which its members, under the guidance and inspiration of Professor Sylvester, have taken in the development of the great geometrical theory of link-work movements.

> "Verum hæc ipse equidem spatiis exclusus iniquis
> Prætereo, atque aliis post commemoranda relinquo."

I will not sit down without again offering my excuses for the fragmentary and disconnected nature of the reflexions I have laid before you this evening. But, indeed, over so wide a field I could only take a wandering course. My object has been to impress upon those who have been indulgent enough to listen to me that the vast increase which this century has witnessed in the extent of the ground already covered by mathematical science has been accompanied by a proportionate increase in the number and variety of the objects of interest to which the mathematician may turn his attention, and by an even more than proportionate increase in the opportunities of discovering new truths which have been brought within his reach. Our border on every side is the unknown; and the further our boundary line is extended, the more multitudinous become the points at which we may hope to penetrate beyond it. In these days when so much is said of original research and of the advancement of scientific knowledge, I feel that it is the business of our Society to see that, so far as our own country is concerned, mathematical science should still be in the vanguard of progress. I should not wish to use words which may seem to reach too far, but I often find the conviction forced upon me that the increase of mathematical knowledge is a necessary condition for the advancement of science, and, if so, a no less necessary condition for the improvement of mankind. I could not augur well for the enduring intellectual strength of any nation of men, whose education was not based on a solid foundation of mathematical learning, and whose scientific conceptions, or in other words whose notions of the world and of the things in it, were not braced and girt together with a strong framework of mathematical reasoning. It is something for men to learn what proof is, and what it is not; and I do not know where this lesson can be better learned than in the schools of a science which has never had to take one footstep backward, which has never asserted without proof, or retracted a proved assertion; a science which, while ever advancing with human civilization, is as unchangeable in its principles as human reason; the same at all times and in all places; so that the work done at Alexandria or Syracuse two thousand years ago (whatever may have been added to it since) is as perfect in its kind, and as direct and unerring in its appeal to our intelligence, as if it had been done yesterday at Berlin or Göttingen by one of our own contemporaries. Perhaps also

it might not be impossible to show, and even from instances within our own time, that a decline in the mathematical productiveness of a people implies a decline in intellectual force along the whole line; and it might not be absurd to contend that on this ground the maintenance of a high standard of mathematical attainment among the scientific men of a country is an object of almost national concern. But I need not ask your assent to such wide assertions; I shall be more than satisfied if anything that may have fallen from me may induce any one of us to think more highly than he has hitherto done of the first and greatest of the sciences, and more hopefully of the part which he himself may bear in its advancement.

On Curves having Four-point Contact with a Triply-infinite Pencil of Curves. *By* W. Spottiswoode, Esq., F.R.S.

[*Read November 9th*, 1876.]

In a paper published in the "Mathematische Annalen" (Vol. iii., p. 459) Brill has investigated the case of curves having three-point contact with a doubly-infinite pencil of curves; and in the same journal (Vol. x., p. 221) H. Krey of Kiel has applied a method, similar to that of Brill, to the next step, viz., the problem now proposed. He does not, however, appear to have succeeded in completely eliminating the differentials which occur in the process; and in that respect his solution is incomplete. Some formulæ, however, used in my papers on the contact of curves and of surfaces, and in particular in that "On the Sextactic Points of a Plane Curve" (Phil. Trans., 1865, p. 657), prove to be directly applicable to the question. An application of them to Brill's problem will be found in a paper in the "Comptes Rendus," 1876 (2e sémestre, p. 627).

In fact, if we put H for the Hessian of U; and if

$$\Delta = \begin{vmatrix} u_1, & w', & v', & d_x \\ w', & v_1, & u', & d_y \\ v', & u', & w_1, & d_z \\ d_x, & d_y, & d_z, & . \end{vmatrix} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})\,(d_x, d_y, d_z)^2,$$

where u, v, w are the first, and u_1, v_1, w_1, u', v', w' the second differential coefficients of U; or writing, as is sometimes convenient,

$$\mathrm{A} = \mathfrak{A}d_x + \mathfrak{H}d_y + \mathfrak{G}d_z \qquad \Delta = \mathrm{A}d_x + \mathrm{B}d_y + \mathrm{C}d_z$$
$$\mathrm{B} = \mathfrak{H}d_x + \mathfrak{B}d_y + \mathfrak{F}d_z$$
$$\mathrm{C} = \mathfrak{G}d_x + \mathfrak{F}d_y + \mathfrak{C}d_z;$$

and further, if Δ' represent any one of the operations

$$u\Delta - \varpi \mathrm{H} d_x,$$
$$v\Delta - \varpi \mathrm{H} d_y,$$
$$w\Delta - \varpi \mathrm{H} d_z,$$

and $$\varpi = 1 + 2(m-1) : (n-1);$$

then the four conditions for four-point contact between two curves U and V, of the degrees n and m respectively, may be written thus:

$$u : v : w = d_x \mathrm{V} : d_y \mathrm{V} : d_z \mathrm{V} = d_x \Delta' \mathrm{V} : d_y \Delta' \mathrm{V} : d_z \Delta' \mathrm{V} \ldots\ldots (1).$$

Now consider four curves of the degree m, $\phi=0$, $\psi=0$, $\chi=0$, $\omega=0$; and the pencil

$$\mathrm{V} = \alpha\phi + \beta\psi + \gamma\chi + \delta\omega = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

then, introducing two indeterminate quantities ϵ, ϵ', the conditions that the curve U may have four-point contact with any one curve of the pencil V may be written as follows:

$$\left.\begin{array}{ccccc} \alpha d_x\phi + \beta d_x\psi + \gamma d_x\chi + \delta d_x\omega + \epsilon u = 0 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ \alpha d_x\Delta'\phi + \beta d_x\Delta'\psi + \gamma d_x\Delta'\chi + \delta d_x\Delta'\omega + \epsilon' u = 0 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \end{array}\right\} \ldots\ldots\ldots (3).$$

But from these six equations we may eliminate the six quantities α, β, ... ϵ', and obtain a resultant independent of them, that is to say, appertaining not to any particular curve of the pencil V, but to the entire pencil. This resultant Φ will consequently represent a curve passing through all the points at which U has four-point contact with any curve of the pencil V. The expression for Φ is

$$\Phi = \begin{vmatrix} d_x\phi & d_x\psi & d_x\chi & d_x\omega & u & . \\ d_y\phi & d_y\psi & d_y\chi & d_y\omega & v & . \\ d_z\phi & d_z\psi & d_z\chi & d_z\omega & w & . \\ d_x\Delta'\phi & d_x\Delta'\psi & d_x\Delta'\chi & d_x\Delta'\omega & . & u \\ d_y\Delta'\phi & d_y\Delta'\psi & d_y\Delta'\chi & d_y\Delta'\omega & . & v \\ d_z\Delta'\phi & d_z\Delta'\psi & d_z\Delta'\chi & d_z\Delta'\omega & . & w \end{vmatrix} = 0 \ldots\ldots (4).$$

The degree of this expression is apparently $4m+8n-20$; but if we take the form $\Delta' = u\Delta - \varpi \mathrm{H} d_x$ it is divisible by u, as is easily seen by writing for line 4, x (line 4) $+ y$ (line 5) $+ z$ (line 6) $- \varpi$H (line 1). The degree is thus reduced to $4m+7n-19$.

This being so, let $m_1 = m+3n-8 =$ degree of $\Delta'\phi$; and let

$$\mu = (m-n) : m_1, \quad \therefore \; n : m = (m+n-m) : m = 1+\mu;$$
$$\mu_1 = (n-m_1) : m_1, \qquad n : m_1 = (m_1+n-m_1) : m_1 = 1+\mu_1.$$

It will then be found that

$$x^2\Phi : mm_1 = \begin{vmatrix} \phi & \ldots\ldots & U+\mu U & . \\ d_y\phi & \ldots\ldots & v & . \\ d_z\phi & \ldots\ldots & w & . \\ \Delta'\phi & \ldots\ldots & . & U+\mu_1 U \\ d_y\Delta'\phi & \ldots\ldots & . & v \\ d_z\Delta'\phi & \ldots\ldots & . & w \end{vmatrix}$$

and if $[\mu U]$, $[\mu_1 U]$, $[\mu U \,.\, \mu_1 U]$, represent the coefficients of μU, $\mu_1 U$, $\mu\mu_1 U^2$, respectively, in the development of (5), we may put

$$x^2\Phi : mm_1 = PU^2 + QU + R \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6),$$

where $P = [\mu U \,.\, \mu_1 U]$, $Q = [\mu U] + [\mu_1 U]$, and R is a function, the form of which is obvious, and whose properties will be noticed in the sequel.

It is proposed to examine the nature of the contact between Φ and U at the points (if any) where ϕ, ψ, χ, ω, U all meet; *i.e.*, where $\phi=0$, $\psi=0$, $\chi=0$, $\omega=0$, $U=0$. For this purpose, if we differentiate (6) with respect to x, we obtain

$$(2x\Phi + x^2 d_x\Phi) : mm_1 = 2PUu + U^2 d_x P + Qu + U d_x Q + d_x R,$$

or, since Φ and U are both supposed to vanish at the points in question, we obtain the system

$$\left.\begin{aligned} x^2 d_x\Phi : mm_1 &= Qu + d_x R \\ x^2 d_y\Phi : mm_1 &= Qv + d_y R \\ x^2 d_z\Phi : mm_1 &= Qw + d_z R \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

But

$$R = \begin{vmatrix} \phi \ldots U & . \\ d_y\phi \ldots v & . \\ d_z\phi \ldots w & . \\ \vdots \quad\quad \vdots & \vdots \end{vmatrix}; \quad \therefore\ d_x R = \begin{vmatrix} d_z\phi \ldots u & . \\ d_y\phi \ldots v & . \\ d_z\phi \ldots w & . \\ \vdots \quad\quad \vdots & \vdots \end{vmatrix} + \text{terms which vanish with } \phi, \psi, \chi, \omega, U.$$

Hence

$$x d_x R = \begin{vmatrix} m\phi \ldots nU & . \\ d_y\phi \ldots v & . \\ d_z\phi \ldots w & . \end{vmatrix} = 0 \text{ at the points in question.}$$

Also, it will be found by a similar process that at the points in question $d_y R$ and $d_z R$ vanish identically. Hence the system (7) becomes

$$d_x\Phi : u = d_y\Phi : v = d_z\Phi : w = mm_1 Q : x^2 \quad \ldots\ldots\ldots\ldots \quad (8),$$

which are in fact the conditions that Φ and U may have an ordinary contact at the points (if any) where $\phi=0$, ... $U=0$.

Again, operating with Δ upon (6), and remembering that from the nature of the operation Δ, if p, q be any functions of x, y, z, $\Delta pq = p\Delta q + q\Delta p + 2\,(\mathfrak{A} \ldots)(d_z p, \ldots)(d_z q, \ldots)$; we obtain

$$(2\mathfrak{A}\Phi + x^2\Delta\Phi + 4x\Delta\Phi) : mm_1 = \Delta(PU^2) + \Delta(QU) + \Delta R.$$

But $$\Delta U = 3H, \quad \Delta U^2 = 2U\Delta U + 2(\mathfrak{A}, \ldots)(u, \ldots)^2$$
$$= 6UH + 2n(n-1)^{-1}UH$$
$$= 2(4n-3)(n-1)^{-1}UH.$$

Hence, for all points on the curve U, $\Delta(PU^2) = 0$. Also

$$\Delta(QU) = U\Delta Q + 3QH + 2\nu(n-1)^{-1}QH,$$

where ν represents the degree of Q. Further, substituting from (8), $4x\Delta\Phi : mm_1 = 4(n-1)^{-1}QH$; and consequently

$$x^2\Delta\Phi : mm_1 = (3n-3+2\nu-4)(n-1)^{-1}QH + \Delta R.$$

But referring to (5), if N be the degree of Φ, $N+2 = \nu+n$, or $2\nu+2n-4 = 2N$. Hence

$$x^2\Delta\Phi : mm_1 = \left.\begin{aligned} &\{1+2(N-1) : (n-1)\}QH + \Delta R \\ &= \varpi_1 QH + \Delta R \end{aligned}\right\} \ldots\ldots\ldots (9),$$

where ϖ_1 stands in the same relation to Φ that ϖ does to V. Lastly, writing down only the first line of the determinants, where no ambiguity is likely to arise, and omitting terms which obviously vanish, we obtain

$$\Delta R = \begin{vmatrix} \Delta\phi & d_x\phi \\ d_y\phi & d_y\phi \\ \vdots & \vdots \end{vmatrix} + 2\begin{vmatrix} d_x\phi & d_y\phi \\ d_y\phi & d_y\phi \\ \vdots & \vdots \end{vmatrix} A + 2\begin{vmatrix} d_y\phi & d_z\phi \\ d_y\phi & d_y\phi \\ \vdots & \vdots \end{vmatrix} B + 2\begin{vmatrix} d_z\phi & \\ d_y\phi & \\ \vdots & \end{vmatrix} C$$
$$= \begin{vmatrix} \Delta\phi & d_x\phi \\ d_y\phi & d_y\phi \\ \vdots & \vdots \end{vmatrix} + 2A\begin{vmatrix} d_x\phi & d_x\phi \\ d_y\phi & d_y\phi \\ \vdots & \vdots \end{vmatrix} + B\begin{vmatrix} d_y\phi & d_y\phi \\ d_y\phi & d_y\phi \\ \vdots & \vdots \end{vmatrix} + C\begin{vmatrix} d_z\phi & \\ d_y\phi & \\ \vdots & \end{vmatrix} = -\begin{vmatrix} \Delta\phi & d_x\phi \\ d_y\phi & d_y\phi \\ \vdots & \vdots \end{vmatrix} + 2A\begin{vmatrix} d_x\phi & \\ d_y\phi & \\ \vdots & \end{vmatrix} \ldots (10).$$

and

$$\begin{vmatrix} \Delta\phi & \Delta\phi & \ldots & 3H \\ d_y\phi & d_y\phi & \ldots & v \\ \vdots & \vdots & & \vdots \end{vmatrix} = +(3-\varpi)QH,$$

$$\begin{vmatrix} Ad_x\phi & Ax^{-1}m\phi & \ldots & U+\mu U \\ d_y\phi & d_y\phi & \ldots & v \\ \vdots & \vdots & & \vdots \end{vmatrix} = x^{-1}Am\begin{vmatrix} d_x\phi & \\ d_y\phi & \\ \vdots & \end{vmatrix} + m\mu(n-1)^{-1}HQ$$

Hence $$\Delta R = (-2n+2+2m-2+2n-2m)QH$$
$$= 0.$$

And consequently we have finally

$$d_x\Phi : u = d_y\Phi : v = d_z\Phi : w = \Delta\Phi : \varpi_1 H \ldots\ldots\ldots (11),$$

which proves that at the points in question, Φ has a three-point contact with U.

[P.S. 16th Dec., 1876.—If the degree of U be the same as that of $\phi, \psi, \ldots$; *i.e.*, if $m = n$, then $P = 0$, $Q = [\mu_1 U]$. But in that case Q vanishes with $\phi, \psi, \ldots$; and consequently

$$d_x\Phi = 0, \quad d_y\Phi = 0, \quad d_z\Phi = 0, \quad \Delta\Phi = 0,$$

so that in this case the curve Φ has a node at the points in question.]

We may also examine the nature of the points on Φ which coincide with such double points of U (if any) as are common also to the curves $\phi, \dots w$. At these points $u=0$, $v=0$, $w=0$, $\mathrm{H}=0$; and consequently, from (11),

$$d_x\Phi=0, \quad d_y\Phi=0, \quad d_z\Phi=0, \quad \Delta\Phi=0, \quad \mathrm{H}(\Phi)=0.$$

The points are consequently also double points of Φ. But, in order to ascertain the values of the differential coefficients of $\Delta\Phi$ at these points, we must find the full expression for $\Delta\Phi$, instead of that given in (9). For this purpose, if ν' be the degree of P, we have

$$\begin{aligned} \mathrm{N}+2 &= \nu'-2(n-1), \\ \nu' &= \mathrm{N}+2n \\ &= 4m+10n-20, \end{aligned}$$

$$\begin{aligned} \Delta(\mathrm{PU}^2) &= \mathrm{U}^2\Delta\mathrm{P}+\mathrm{P}\Delta(\mathrm{U}^2)+2(4m+10n-20)(n-1)^{-1}\,\mathrm{UPH} \\ &= \mathrm{U}^2\Delta\mathrm{P}+2(4n-3)(n-1)^{-1}\,\mathrm{UPH} \\ &\qquad +2(4m+10n-20)(n-1)^{-1}\,\mathrm{UPH} \\ &= \mathrm{U}^2\Delta\mathrm{P}+2(4m+14n-23)(n-1)^{-1}\,\mathrm{UPH} \\ \Delta(\mathrm{QU}) &= \mathrm{U}\Delta\mathrm{Q}+3\mathrm{QH}+2(4m+7n-18)(n-1)^{-1}\,\mathrm{QH} \\ &= \mathrm{U}\Delta\mathrm{Q}+(8m+17n-39)(n-1)^{-1}\,\mathrm{QH}. \end{aligned}$$

Hence

$$\begin{aligned} (2\mathfrak{A}\Phi+x^2\Delta\Phi+4x\mathrm{A}\Phi) : mm_1 = \quad & \mathrm{U}^2\Delta\mathrm{P}+\mathrm{U}\Delta\mathrm{Q}+\Delta\mathrm{R} \\ & +2(4m+14n-23)(n-1)^{-1}\,\mathrm{UPH} \\ & +(8m+17n-39)(n-1)^{-1}\,\mathrm{QH}. \end{aligned}$$

Differentiating with respect to x,

$$\begin{aligned} (2d_x\mathfrak{A}\,.\,\Phi+2\mathfrak{A}d_x\Phi+2x\Delta\Phi+2x^2d_x\Delta\Phi+4\mathrm{A}\Phi+4xd_x\mathrm{A}\,.\,\Phi+4x\mathrm{A}d_x\Phi) : mm_1 \\ = \quad \mathrm{U}^2d_x\Delta\mathrm{P}+2\mathrm{U}u\Delta\mathrm{P}+\mathrm{U}d_x\Delta\mathrm{Q}+u\Delta\mathrm{Q}+d_x\Delta\mathrm{R} \\ +2(4m+14n-23)(n-1)^{-1}\,(\mathrm{UP}p+\mathrm{UH}d_x\mathrm{P}+u\mathrm{PH}) \\ +(8m+17n-39)(n-1)^{-1}\,(\mathrm{Q}p+\mathrm{H}d_x\mathrm{Q}), \end{aligned}$$

or, omitting terms which vanish at the point in question,

$$\begin{aligned} (2x^2d_x\Delta\Phi+4xd_x\mathrm{A}\,.\,\Phi+4x\mathrm{A}d_x\Phi) : mm_1 \\ = d_x\Delta\mathrm{R}+(8m+17n-39)(n-1)^{-1}\,\mathrm{Q}p. \end{aligned}$$

But

$$\begin{aligned} d_x\mathrm{A}\,.\,\Phi &= d_x\mathfrak{A}d_x\Phi+d_x\mathfrak{B}d_y\Phi+d_x\mathfrak{C}d_z\Phi=0, \\ \mathrm{A}d_x\Phi &= \mathfrak{A}d_x^2\Phi+\mathfrak{B}d_yd_x\Phi+\mathfrak{C}d_zd_x\Phi; \end{aligned}$$

and, on differentiating (6), we obtain

$$\begin{aligned} (2\Phi+4xd_x\Phi+x^2d_x^2\Phi) : mm_1 = \quad & 2\mathrm{U}ud_x\mathrm{P}+2u^2\mathrm{P}+2\mathrm{P}\mathrm{U}u_1 \\ & +2\mathrm{U}ud_x\mathrm{P}+\mathrm{U}^2d_x^2\mathrm{P} \\ & +ud_x\mathrm{Q}+\mathrm{Q}u_1 \\ & +\mathrm{U}d_x^2\mathrm{Q}+ud_x\mathrm{Q}+d^2\mathrm{R}, \end{aligned}$$

or, omitting terms which vanish,

$$x^2 d_x^2 \Phi : mm_1 = Qu_1 + d_x^2 R,$$
$$x^2 d_y d_x \Phi : mm_1 = Qw' + d_y d_x R,$$
$$x^2 d_z d_x \Phi : mm_1 = Qv' + d_z d_x R.$$

Hence $$x^2 A d_x \Phi : mm_1 = A d_x R.$$

To evaluate the terms in R it will be simplest to differentiate with respect to columns, and, referring to the value of R derived from (5), it is easily seen that the only terms which need be retained in the value of $d_x^2 R$ will be

$$\begin{array}{cccc} \phi & \dots & u & . \\ d_y\phi & \dots & w' & . \\ d_z\phi & \dots & v' & . \\ \Delta'\phi & \dots & . & u \\ d_y\Delta'\phi & \dots & . & w' \\ d_z\Delta'\phi & \dots & . & v' \end{array}$$

which vanishes in virtue of the first line. Operating in the same way for $d_y d_x R$, $d_z d_x R$, we obtain

$$d_x^2 R = 0, \quad d_y d_x R = 0, \quad d_z d_x R = 0; \quad \therefore A d_x R = 0, \quad A d_x \Phi = 0.$$

Operating in a similar manner with Δ, it will be seen that the result must be of the form $L + MH + NH^2$, whereof L contains two (the last two) vanishing columns, M one (the last, or last but one) vanishing column; so that its first differential coefficient with respect to any of the variables must vanish. Hence we may conclude that

$$d_x \Delta R = 0, \quad d_y \Delta R = 0, \quad d_z \Delta R = 0;$$

and consequently

$$4x^2 d_x \Delta \Phi : mm_1 = (8m + 17n - 39)(n-1)^{-1} Qp.$$

But at the points in question $Q = 0$; hence finally

$$d_x \Delta \Phi = 0, \quad d_y \Delta \Phi = 0, \quad d_z \Delta \Phi = 0,$$

and consequently $H(\Delta\Phi) = 0$. That is to say, the curve $\Delta\Phi$, which passes through the double points of Φ, has itself a double point at those points. The points in question (which are also those double points of U through which pass all the curves ϕ, ψ, χ, ω) are consequently quadruple points of Φ.

On some Classes of Multiple Definite Integrals.

By E. B. Elliott, B.A.

[*Read November 9th*, 1876.]

1. In what follows, repeated use will be made of the facts that, a and a' being any positive constants,

$$\int_0^\infty \frac{\phi(ax)-\phi(a'x)}{x}\,dx = \int_{a'}^{a} da \int_0^\infty \phi'(ax)\,dx = \log\frac{a}{a'}\,\{\phi(\infty)-\phi(0)\} \quad \ldots\ldots\ldots(1)$$

if the limits $\phi(\infty)$ and $\phi(0)$ are definite, and that

$$\int_{-\infty}^{0} \frac{\phi(ax)-\phi(a'x)}{x}\,dx = \int_{a'}^{a} da \int_{-\infty}^{0} \phi'(ax)\,dx = \log\frac{a}{a'}\,\{\phi(0)-\phi(-\infty)\} \quad \ldots\ldots\ldots(2)$$

if the limits $\phi(0)$ and $\phi(-\infty)$ are both definite.

Consider first the double integral $\int_0^\infty\int_0^\infty \frac{\phi(ax+by)-\phi(a'x+b'y)}{xy}\,dx\,dy$, in which a, a', b, b' are any positive constants.

By (1), we have

$$\int_0^\infty\int_0^\infty \frac{\phi(ax+by)-\phi(ax+b'y)}{xy}\,dx\,dy = \int_0^\infty \log\frac{b}{b'}\,\{\phi(\infty)-\phi(ax)\}\,\frac{dx}{x}$$

provided that neither $\phi(\infty)$ nor $\phi(ax)$ be infinite, and

$$\int_0^\infty\int_0^\infty \frac{\phi(ax+b'y)-\phi(a'x+b'y)}{xy}\,dx\,dy = \int_0^\infty \log\frac{a}{a'}\,\{\phi(\infty)-\phi(b'y)\}\,\frac{dy}{y}$$

under a like condition. Thus, adding,

$$\int_0^\infty\int_0^\infty \frac{\phi(ax+by)-\phi(a'x+b'y)}{xy}\,dx\,dy = \left\{\log\frac{a}{a'}+\log\frac{b}{b'}\right\}\phi(\infty)\int_0^\infty\frac{dx}{x} - \log\frac{b}{b'}\int_0^\infty\frac{\phi(ax)}{x}\,dx - \log\frac{a}{a'}\int_0^\infty\frac{\phi(b'x)}{x}\,dx.$$

Therefore, in the case when $\log\frac{a}{a'}+\log\frac{b}{b'}=0$, *i.e.*, when $ab=a'b'$,

$$\int_0^\infty\int_0^\infty \frac{\phi(ax+by)-\phi(a'x+b'y)}{xy}\,dx\,dy = \log\frac{a}{a'}\int_0^\infty\frac{\phi(ax)-\phi(b'x)}{x}\,dx = \log\frac{a}{a'}\log\frac{a}{b'}\,\{\phi(\infty)-\phi(0)\} \quad \ldots\ldots\ldots(3),$$

the only restriction as to the nature of the function ϕ being, that it be not made infinite when its argument is zero or any positive quantity finite or infinite.

In precisely the same manner, using (2), we find that, if the function

ϕ be such as not to become infinite when its argument is zero or any negative quantity, then, under the same condition $ab = a'b'$,

$$\int_{-\infty}^{0}\int_{-\infty}^{0} \frac{\phi(ax+by)-\phi(a'x+b'y)}{xy}\,dx\,dy = \log\frac{a}{a'}\log\frac{a}{b'}\{\phi(0)-\phi(-\infty)\} \quad \ldots\ldots\ldots\ldots(4).$$

In the results (3) and (4), it is of course to be noticed that a may be either one of the two constants a and b.

2. Consider now the triple integral

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+by+cz)-\phi(a'x+b'y+c'z)}{xyz}\,dx\,dy\,dz,$$

the function ϕ being of the same nature as in (3), and the coefficients a, b, c, a', b', c' all positive. We have, by three applications of (1),

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+by+cz)-\phi(ax+by+c'z)}{xyz}\,dx\,dy\,dz = \log\frac{c}{c'}\int_0^\infty\int_0^\infty \frac{\phi(\infty)-\phi(ax+by)}{xy}\,dx\,dy,$$

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+by+c'z)-\phi(ax+b'y+c'z)}{xyz}\,dx\,dy\,dz = \log\frac{b}{b'}\int_0^\infty\int_0^\infty \frac{\phi(\infty)-\phi(ax+c'z)}{xz}\,dx\,dz,$$

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+b'y+c'z)-\phi(a'x+b'y+c'z)}{xyz}\,dx\,dy\,dz = \log\frac{a}{a'}\int_0^\infty\int_0^\infty \frac{\phi(\infty)-\phi(b'y+c'z)}{yz}\,dy\,dz.$$

If then $\log\frac{a}{a'}+\log\frac{b}{b'}+\log\frac{c}{c'}=0$, *i.e.*, if $abc = a'b'c'$, we have, by addition,

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+by+cz)-\phi(a'x+b'y+c'z)}{xyz}\,dx\,dy\,dz$$

$$= -\log\frac{c}{c'}\int_0^\infty\int_0^\infty \frac{\phi(ax+by)}{xy}\,dx\,dy+\left(\log\frac{c}{c'}+\log\frac{a}{a'}\right)\int_0^\infty\int_0^\infty \frac{\phi(ax+c'z)}{xz}\,dx\,dz - \log\frac{a}{a'}\int_0^\infty\int_0^\infty \frac{\phi(b'y+c'z)}{yz}\,dy\,dz$$

$$= \log\frac{a}{a'}\int_0^\infty\int_0^\infty \frac{\phi(ax+c'y)-\phi(b'x+c'y)}{xy}\,dx\,dy - \log\frac{c}{c'}\int_0^\infty\int_0^\infty \frac{\phi(ax+by)-\phi(ax+c'y)}{xy}\,dx\,dy$$

$$= \log\frac{a}{a'}\log\frac{a}{b'}\int_0^\infty \frac{\phi(\infty)-\phi(c'y)}{y}\,dy - \log\frac{c}{c'}\log\frac{b}{c'}\int_0^\infty \frac{\phi(\infty)-\phi(ax)}{x}\,dx, \text{ by (1)};$$

and so, provided that $\log\frac{a}{a'}\log\frac{a}{b'} = \log\frac{c}{c'}\log\frac{b}{c'}$,

$$= \log\frac{a}{a'}\log\frac{a}{b'}\int_0^\infty \frac{\phi(ax)-\phi(c'x)}{x}\,dx$$

$$= \log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\{\phi(\infty)-\phi(0)\} \quad \ldots\ldots(5).$$

The two conditions in the six constants formed above for the truth of this may be written

$$abc = a'b'c', \quad \log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'} = \log\frac{a}{c'}\log\frac{b}{c'}\log\frac{c}{c'},$$

In precisely the same manner, using (2), we find that, ϕ being of the same nature as in (4), under the same two conditions in the coefficients,

$$\int_{-\infty}^0\int_{-\infty}^0\int_{-\infty}^0 \frac{\phi(ax+by+cz)-\phi(a'x+b'y+c'z)}{xyz}\,dx\,dy\,dz$$

$$= \log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\{\phi(0)-\phi(-\infty)\} \quad \ldots\ldots(6).$$

It is to be noticed that, in these conditions and results, a denotes any one of the three a, b, c, and c' any one of a', b', c'.

3. Again, the quadruple integral, in which ϕ is of the same nature as in (3) and (5), and the constants a, b, c, d, a', b', c', d' all positive,

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+by+cz+du)-\phi(a'x+b'y+c'z+d'u)}{xyzu}\,dx\,dy\,dz\,du$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+by+cz+du)-\phi(ax+by+c'z+d'u)}{xyzu}\,dx\,dy\,dz\,du$$

$$+\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+by+c'z+d'u)-\phi(a'x+b'y+c'z+d'u)}{xyzu}\,dx\,dy\,dz\,du$$

$$= \log\frac{c}{c'}\log\frac{c}{d'}\int_0^\infty\int_0^\infty \frac{\phi(\infty)-\phi(ax+by)}{xy}\,dx\,dy$$

$$+\log\frac{a}{a'}\log\frac{a}{b'}\int_0^\infty\int_0^\infty \frac{\phi(\infty)-\phi(c'z+d'u)}{zu}\,dz\,du,$$

by (3), provided that $ab = a'b'$ and $cd = c'd'$. If, in addition, $\log\frac{a}{a'}\log\frac{a}{b'} + \log\frac{c}{c'}\log\frac{c}{d'} = 0$, this becomes

$$= \log\frac{a}{a'}\log\frac{a}{b'}\int_0^\infty\int_0^\infty \frac{\phi(ax+by)-\phi(c'x+d'y)}{xy}\,dx\,dy$$

$$= \log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\{\phi(\infty)-\phi(0)\} \quad \ldots\ldots(7),$$

if the additional condition $ab = c'd'$ is satisfied. Thus there are alto-

gether four conditions

$$ab = cd = a'b' = c'd', \quad \log\frac{a}{a'}\log\frac{a}{b'} + \log\frac{c}{c'}\log\frac{c}{d'} = 0.$$

In like manner, under the same four conditions, ϕ being now of the same nature as in (4) and (6),

$$\int_{-\infty}^{0}\int_{-\infty}^{0}\int_{-\infty}^{0}\int_{-\infty}^{0} \frac{\phi(ax+by+cz+du)-\phi(a'x+b'y+c'z+d'u)}{xyzu}\,dx\,dy\,dz\,du$$
$$= \log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\{\phi(0)-\phi(-\infty)\} \quad \ldots\ldots(8).$$

Like methods may now be employed to calculate the values of the analogous multiple integrals of higher orders under conditions which are obtained in the course of the work.

4. Again, consider the double integral

$$\int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(by)-\phi(a'x)\,\phi(b'y)}{xy}\,dx\,dy,$$

the function ϕ being such as in (3), (5), and (7). We have

$$\int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(by)-\phi(ax)\,\phi(b'y)}{xy}\,dx\,dy$$
$$= \int_0^\infty \frac{\phi(ax)}{x}\,dx\int_0^\infty \frac{\phi(by)-\phi(b'y)}{y}\,dy$$
$$= \log\frac{b}{b'}\{\phi(\infty)-\phi(0)\}\int_0^\infty \frac{\phi(ax)}{x}\,dx, \text{ by (1)},$$

and

$$\int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(b'y)-\phi(a'x)\,\phi(b'y)}{xy}\,dx\,dy$$
$$= \log\frac{a}{a'}\{\phi(\infty)-\phi(0)\}\int_0^\infty \frac{\phi(b'y)}{y}\,dy$$

in like manner. Thus, adding, under the condition $\log\frac{a}{a'} + \log\frac{b}{b'} = 0$, *i.e.*, $ab = a'b'$, we have that in that case

$$\int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(by)-\phi(a'x)\,\phi(b'y)}{xy}\,dx\,dy$$
$$= -\log\frac{a}{a'}\{\phi(\infty)-\phi(0)\}\int_0^\infty \frac{\phi(ax)-\phi(b'x)}{x}\,dx$$
$$= -\log\frac{a}{a'}\log\frac{a}{b'}\{\phi(\infty)-\phi(0)\}^2 \quad \ldots\ldots(9).$$

In precisely the same way, by (2),

$$\int_{-\infty}^{0}\int_{-\infty}^{0} \frac{\phi(ax)\,\phi(by)-\phi(a'x)\,\phi(b'y)}{xy}\,dx\,dy$$
$$= -\log\frac{a}{a'}\log\frac{a}{b'}\{\phi(0)-\phi(-\infty)\}^2 \quad \ldots\ldots(10),$$

under the same condition $ab = a'b'$, ϕ in this case being of the same nature as in (4), (6), and (8).

5. The triple integral

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(by)\,\phi(cz) - \phi(a'x)\,\phi(b'y)\,\phi(c'z)}{xyz}\,dx\,dy\,dz$$

$$= \int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(by)}{xy}\,dx\,dy \int_0^\infty \frac{\phi(cz) - \phi(c'z)}{z}\,dz$$

$$+ \int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(c'z)}{xz}\,dx\,dz \int_0^\infty \frac{\phi(by) - \phi(b'y)}{y}\,dy$$

$$+ \int_0^\infty\int_0^\infty \frac{\phi(b'y)\,\phi(c'z)}{yz}\,dy\,dz \int_0^\infty \frac{\phi(ax) - \phi(a'x)}{x}\,dx$$

$$= \{\phi(\infty) - \phi(0)\}\left\{\log\frac{c}{c'}\int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(by)}{xy}\,dx\,dy\right.$$

$$\left. + \log\frac{b}{b'}\int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(c'z)}{xz}\,dx\,dz + \log\frac{a}{a'}\int_0^\infty\int_0^\infty \frac{\phi(b'y)\,\phi(c'z)}{yz}\,dy\,dz\right\},$$

which, if $\log\frac{a}{a'} + \log\frac{b}{b} + \log\frac{c}{c'} = 0$, *i.e.*, if $abc = a'b'c'$, may be written

$$= \{\phi(\infty) - \phi(0)\}\left\{\log\frac{a}{c'}\int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(by) - \phi(ax)\,\phi(c'y)}{xy}\,dx\,dy\right.$$

$$\left. - \log\frac{a}{a'}\int_0^\infty\int_0^\infty \frac{\phi(ax)\,\phi(c'y) - \phi(b'x)\,\phi(c'y)}{xy}\,dx\,dy\right\}$$

$$= \{\phi(\infty) - \phi(0)\}^2\left\{\log\frac{c}{c'}\log\frac{b}{c'}\int_0^\infty \frac{\phi(ax)}{x}\,dx\right.$$

$$\left. - \log\frac{a}{a'}\log\frac{a}{b'}\int_0^\infty \frac{\phi(c'y)}{y}\,dy\right\},$$

which, if $\log\frac{a}{a'}\log\frac{c}{b'} = \log\frac{c}{c'}\log\frac{b}{c'}$, may be written

$$= \{\phi(\infty) - \phi(0)\}^2 \log\frac{a}{a'}\log\frac{a}{b'}\int_0^\infty \frac{\phi(ax) - \phi(c'x)}{x}\,dx$$

$$= \log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\{\phi(\infty) - \phi(0)\}^3 \ldots\ldots\ldots\ldots\ldots\ldots \quad (11).$$

So, too, under the same two conditions,

$$abc = a'b'c', \quad \log\frac{a}{a'}\log\frac{a}{b'} = \log\frac{b}{c'}\log\frac{c}{c'},$$

which latter may be written more symmetrically

$$\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'} = \log\frac{a}{c'}\log\frac{b}{c'}\log\frac{c}{c'},$$

$$\int_{-\infty}^{0}\int_{-\infty}^{0}\int_{-\infty}^{0}\frac{\phi(ax)\,\phi(by)\,\phi(cz)-\phi(a'x)\,\phi(b'y)\,\phi(c'z)}{xyz}\,dx\,dy\,dz$$
$$=\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\{\phi(0)-\phi(-\infty)\}^3\;..\;.....\;(12).$$

6. The corresponding quadruple integral

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\frac{\phi(ax)\,\phi(by)\,\phi(cz)\,\phi(du)-\phi(a'x)\,\phi(b'y)\,\phi(c'z)\,\phi(d'u)}{xyzu}\,dx\,dy\,dz\,du$$
$$=\int_0^\infty\int_0^\infty\frac{\phi(ax)\,\phi(by)}{xy}\,dx\,dy\int_0^\infty\int_0^\infty\frac{\phi(cz)\,\phi(du)-\phi(c'z)\,\phi(d'u)}{zu}\,dz\,du$$
$$+\int_0^\infty\int_0^\infty\frac{\phi(c'z)\,\phi(d'u)}{zu}\,dz\,du\int_0^\infty\int_0^\infty\frac{\phi(ax)\,\phi(by)-\phi(a'x)\,\phi(b'y)}{xy}\,dx\,dy$$
$$=-\int_0^\infty\int_0^\infty\frac{\phi(ax)\,\phi(by)}{xy}\,dx\,dy\,.\log\frac{c}{c'}\log\frac{c}{d'}\{\phi(\infty)-\phi(0)\}^2$$
$$-\int_0^\infty\int_0^\infty\frac{\phi(c'z)\,\phi(d'u)}{zu}\,dz\,du\,.\log\frac{a}{a'}\log\frac{a}{b'}\{\phi(\infty)-\phi(0)\}^2,$$

by (9), provided that $cd=c'd'$ and $ab=a'b'$. If in addition

$$\log\frac{a}{a'}\log\frac{a}{b'}+\log\frac{c}{c'}\log\frac{c}{d'}=0,$$

this becomes

$$=\log\frac{a}{a'}\log\frac{a}{b'}\{\phi(\infty)-\phi(0)\}^2\int_0^\infty\int_0^\infty\frac{\phi(ax)\,\phi(by)-\phi(c'x)\,\phi(d'y)}{xy}\,dx\,dy$$
$$=-\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\{\phi(\infty)-\phi(0)\}^4\;.....\;(13),$$

provided that also $ab=c'd'$.

Similarly, under the same four conditions $ab=cd=a'b'=c'd'$ and $\log\frac{a}{a'}\log\frac{a}{b'}+\log\frac{c}{c'}\log\frac{c}{d'}=0$, the value of the quadruple integral of the like expression, each integration being between lower limit $-\infty$ and upper limit 0, is

$$-\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\{\phi(0)-\phi(-\infty)\}^2\;.....\;(14).$$

Like methods may now be used to evaluate the corresponding multiple integrals of higher orders, under conditions in the constants obtained in the course of the work.

It is to be noticed that the conditions for the truth of the theorems (9) to (14) are in each case precisely the same as for that of the corresponding ones of the class (3) to (8).

7. Consider now the quadruple integral

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+by)\,\phi(cz+du)-\phi(a'x+b'y)\,\phi(c'z+d'u)}{xyzu}\,dx\,dy\,dz\,du.$$

It may be written

$$= \int_0^\infty\int_0^\infty \frac{\phi(ax+by)}{xy}\,dx\,dy\int_0^\infty\int_0^\infty \frac{\phi(cz+du)-\phi(c'z+d'u)}{zu}\,dz\,du$$

$$+\int_0^\infty\int_0^\infty \frac{\phi(c'z+d'u)}{zu}\,dz\,du\int_0^\infty\int_0^\infty \frac{\phi(ax+by)-\phi(a'x+b'y)}{xy}\,dx\,dy$$

$$= \int_0^\infty\int_0^\infty \frac{\phi(ax+by)}{xy}\,dx\,dy\,.\,\log\frac{c}{c'}\log\frac{c}{d}\,\{\phi(\infty)-\phi(0)\}$$

$$+\int_0^\infty\int_0^\infty \frac{\phi(c'z+d'u)}{zu}\,dz\,du\,.\,\log\frac{a}{a'}\log\frac{a}{b'}\,\{\phi(\infty)-\phi(0)\},$$

by (3), provided that $cd=c'd'$ and $ab=a'b'$. If also

$$\log\frac{c}{c'}\log\frac{c}{d'}+\log\frac{a}{a'}\log\frac{a}{b'}=0,$$

this becomes

$$-\log\frac{a}{a'}\log\frac{a}{b'}\,\{\phi(\infty)-\phi(0)\}\int_0^\infty\int_0^\infty \frac{\phi(ax+by)-\phi(c'x+d'y)}{xy}\,dx\,dy$$

$$=-\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\,\{\phi(\infty)-\phi(0)\}^2\ldots\ldots\ldots\ldots(15),$$

provided that also $ab=c'd'$.

In just the same way, under the same four conditions in the constants, viz.,

$$ab=cd=a'b'=c'd',\quad\text{and}\quad\log\frac{a}{a'}\log\frac{a}{b'}+\log\frac{c}{c'}\log\frac{c}{d'}=0,$$

$$\int_{-\infty}^0\int_{-\infty}^0\int_{-\infty}^0\int_{-\infty}^0 \frac{\phi(ax+by)\,\phi(cz+du)-\phi(a'x+b'y)\,\phi(c'z+d'u)}{xyzu}\,dx\,dy\,dz\,du$$

$$=-\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\,\{\phi(0)-\phi(-\infty)\}^2\ldots\ldots(16).$$

Observe that the conditions in the constants for the truth of (15) and (16) are the same as for that of the allied identities (7) and (8), and (13) and (14).

8. To take one more example of the same class: by three applications of (3),

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{\phi(ax+by)\,\phi(cz+du)\,\phi(ev+fw) - \phi(a'x+b'y)\,\phi(c'z+d'u)\,\phi(e'v+f'w)}{xyzuvw} \times dx\,dy\,dz\,du\,dv\,dw$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+by)\,\phi(cz+du)}{xyzu}\,dx\,dy\,dz\,du \times \log\frac{e}{e'}\log\frac{e}{f'}\{\phi(\infty)-\phi(0)\}$$

$$+ \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+by)\,\phi(e'v+f'w)}{xyvw}\,dx\,dy\,dv\,dw \times \log\frac{c}{c'}\log\frac{c}{d'}\{\phi(\infty)-\phi(0)\}$$

$$+ \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(c'z+d'u)\,\phi(e'v+f'w)}{zuvw}\,dz\,du\,dv\,dw \times \log\frac{a}{a'}\log\frac{a}{b'}\{\phi(\infty)-\phi(0)\},$$

provided that $ef = e'f'$, $cd = c'd'$, and $ab = a'b'$. If then, in addition,

$$\log\frac{a}{a'}\log\frac{a}{b'} + \log\frac{c}{c'}\log\frac{c}{d'} + \log\frac{e}{e'}\log\frac{e}{f'} = 0,$$

this becomes, by two more applications of (3),

$$= \log\frac{e}{e'}\log\frac{e}{f'}\{\phi(\infty)-\phi(0)\}\int_0^\infty\int_0^\infty \frac{\phi(ax+by)}{xy}\,dx\,dy \times \log\frac{c}{e'}\log\frac{c}{f'}\{\phi(\infty)-\phi(0)\}$$

$$- \log\frac{a}{a'}\log\frac{a}{b'}\{\phi(\infty)-\phi(0)\}\int_0^\infty\int_0^\infty \frac{\phi(e'v+f'w)}{vw}\,dv\,dw \times \log\frac{a}{c'}\log\frac{a}{d'}\{\phi(\infty)-\phi(0)\},$$

provided that also $cd = e'f'$ and $ab = c'd'$,

$$= \log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\{\phi(\infty)-\phi(0)\}^2 \times \int_0^\infty\int_0^\infty \frac{\phi(ax+by)-\phi(e'x+f'y)}{xy}\,dx\,dy,$$

if the additional condition

$$\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'} = \log\frac{c}{e'}\log\frac{c}{f'}\log\frac{e}{e'}\log\frac{e}{f'} \text{ holds,}$$

$$= \log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'}\{\phi(\infty)-\phi(0)\}^3 \ldots (17),$$

if also $ab = e'f'$.

In the same way the value of the like integral, with $-\infty$ and 0 for the limits of each integration, is the same multiple of $\{\phi(0)-\phi(-\infty)\}^3$ under the same conditions, which may be written as seven, thus

$$ab = cd = ef = a'b' = c'd' = e'f',$$

$$\log\frac{a}{a'}\log\frac{a}{b'}+\log\frac{c}{c'}\log\frac{c}{d'}+\log\frac{e}{e}\log\frac{e}{f'} = 0,$$

$$\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'}$$
$$+\log\frac{a}{f'}\log\frac{b}{f'}\log\frac{c}{f'}\log\frac{d}{f'}\log\frac{e}{f'}\log\frac{f}{f'} = 0.$$

9. Take, lastly, the simplest example of another class,

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty \frac{\phi(ax+by+cz)\phi(du+ev+fw)-\phi(a'x+b'y+c'z)\phi(d'u+e'v+f'w)}{xyzuvw}$$
$$\times\, dx\,dy\,dz\,du\,dv\,dw.$$

By two applications of (5), this may be written

$$=\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+by+cz)}{xyz}\,dx\,dy\,dz\,.\log\frac{d}{d'}\log\frac{d}{e'}\log\frac{d}{f'}\{\phi(\infty)-\phi(0)\}$$

$$+\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(d'u+e'v+f'w)}{uvw}\,du\,dv\,dw\,.\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\{\phi(\infty)-\phi(0)\},$$

subject to the four conditions

$$abc = a'b'c',\quad def = d'e'f',\quad \log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'} = \log\frac{a}{a'}\log\frac{b}{a'}\log\frac{c}{a'},$$

$$\log\frac{d}{d'}\log\frac{d}{e'}\log\frac{d}{f'} = \log\frac{d}{d'}\log\frac{e}{d'}\log\frac{f}{d'};$$

$$=-\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\{\phi(\infty)-\phi(0)\}$$
$$\times\int_0^\infty\int_0^\infty\int_0^\infty \frac{\phi(ax+by+cz)-\phi(d'x+e'y+f'z)}{xyz}\,dx\,dy\,dz,$$

if in addition

$$\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}+\log\frac{d}{d'}\log\frac{d}{e'}\log\frac{d}{f'} = 0;$$

$$=-\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'}\{\phi(\infty)-\phi(0)\}^2 \quad \ldots\ (18),$$

if the two other conditions

$$abc = d'e'f',\quad \log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'} = \log\frac{a}{d'}\log\frac{b}{d'}\log\frac{c}{d'}\ \text{hold.}$$

Similarly, under the same seven conditions, the like integral, with all the lower limits $-\infty$ and all the upper ones 0, is the same multiple of $\{\phi(0)-\phi(-\infty)\}^2$.

10. The above theorems may be multiplied by easy transformations. Thus, in (3), the substitution of $\log x$, $\log y$ for x, y respectively gives that, if $ab = a'b'$,

$$\int_1^\infty\int_1^\infty \frac{\phi[\log(x^a y^b)]-\phi[\log(x^{a'}y^{b'})]}{xy\log x\log y}\,dx\,dy = \log\frac{a}{a'}\log\frac{a}{b'}\{\phi(\infty)-\phi(0)\},$$

which may be written

$$\int_1^\infty\int_1^\infty \frac{\psi(x^a y^b)-\psi(x^{a'}y^{b'})}{xy\log x\log y}\,dx\,dy = \log\frac{a}{a'}\log\frac{a}{b'}\{\psi(\infty)-\psi(1)\},$$

the function ψ being such as not to become infinite for any value of its argument equal to or greater than unity.

So (4) gives that, under the same conditions $ab = a'b'$, if ψ be a function not made infinite by any value of its argument between 0 and 1 inclusively,

$$\int_0^1\int_0^1 \frac{\psi(x^a y^b)-\psi(x^{a'}y^{b'})}{xy\log x\log y}\,dx\,dy = \log\frac{a}{a'}\log\frac{a}{b'}\{\psi(1)-\psi(0)\}.$$

The same substitutions applied to (9) and (10) give

$$\int_1^\infty\int_1^\infty \frac{\psi(x^a)\,\psi(y^b)-\psi(x^{a'})\,\psi(y^{b'})}{xy\log x\log y}\,dx\,dy = -\log\frac{a}{a'}\log\frac{a}{b'}\{\psi(\infty)-\psi(1)\}^2,$$

and

$$\int_0^1\int_0^1 \frac{\psi(x^a)\,\psi(y^b)-\psi(x^{a'})\,\psi(y^{b'})}{xy\log x\log y}\,dx\,dy = -\log\frac{a}{a'}\log\frac{a}{b'}\{\psi(1)-\psi(0)\}^2,$$

under the same condition $ab = a'b'$.

And so like transformations may be applied to all the other above results (5) to (8) and (11) to (18), giving a new series of theorems true under the same conditions as before.

All these results may however be proved independently, using instead of (1) and (2) the equivalent formulæ

$$\int_1^\infty \frac{\psi(x^a)-\psi(x^{a'})}{x\log x}\,dx = \log\frac{a}{a'}\{\psi(\infty)-\psi(1)\},$$

$$\int_0^1 \frac{\psi(x^a)-\psi(x^{a'})}{x\log x}\,dx = \log\frac{a}{a'}\{\psi(1)-\psi(0)\},$$

which may be seen at once just as (1) and (2) were.

11. Again, transform (9) by substituting ϵ^x, ϵ^y for x, y respectively;

and for convenience write also a for $\log a$, β for $\log b$, &c. Then it becomes that, if $a+\beta = a'+\beta'$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{\phi(\epsilon^{x+a})\,\phi(\epsilon^{y+\beta}) - \phi(\epsilon^{x+a'})\,\phi(\epsilon^{y+\beta'})\}\,dx\,dy$$
$$= -(a-a')(a-\beta')\{\phi(\infty)-\phi(0)\}^2,$$

which may be written

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{F(x+a)\,F(y+\beta) - F(x+a')\,F(y+\beta')\}\,dx\,dy$$
$$= -(a-a')(a-\beta')\{F(\infty)-F(-\infty)\}^2,$$

F being a function which no value of its argument makes infinite.

A like transformation applied to (11) gives that, under the conditions

$$a+\beta+\gamma = a'+\beta'+\gamma',$$

$$(a-a')(a-\beta')(a-\gamma') = (a-\gamma')(\beta-\gamma')(\gamma-\gamma'),$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{F(x+a)\,F(y+\beta)\,F(z+\gamma) - F(x+a')\,F(y+\beta')\,F(z+\gamma')\}\,dx\,dy\,dz$$
$$= (a-a')(\beta-\beta')(\gamma-\gamma')\{F(\infty)-F(-\infty)\}^3.$$

In like manner, from (13), if $a+\beta = \gamma+\delta = a'+\beta' = \gamma'+\delta'$, and $(a-a')(a-\beta')+(\gamma-\gamma')(\gamma-\delta') = 0$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{F(x+a)\,F(y+\beta)\,F(z+\gamma)\,F(u+\delta) - F(x+a')\,F(y+\beta')\,F(z+\gamma')\,F(u+\delta')\}\,dx\,dy\,dz\,du$$
$$= -(a-a')(a-\beta')(a-\gamma')(a-\delta')\{F(\infty)-F(-\infty)\}^4.$$

Other though less simply expressed theorems are obtained by similar transformations from (3), (5), (7), (15), (17), (18) above.

All these, however, may be proved easily without transformation. For they can be deduced, in the same way as their corresponding theorems have been from (1), by means of

$$\int_{-\infty}^{\infty} \{F(x+a)-F(x+\beta)\}\,dx = (a-\beta)\{F(\infty)-F(-\infty)\},$$

a transformation of (1) proved readily in the same way.

12. If it be allowable to substitute x^m, y^n, z^p, &c., for x, y, z, &c., in results (3) to (18), much more general results are obtained under the same conditions as before. I have not, however, been able to prove directly any of these theorems except in the special case when $m = n = p =$ &c., and have considerable doubt as to the lawfulness in general of the transformation. The class, however, for which

$m = n =$ &c., m being any positive quantity, of which the first, the tranformation of (3), is

$$\int_0^\infty \int_0^\infty \frac{\phi(ax^m + by^m) - \phi(a'x^m + b'y^m)}{xy}\, dx\, dy$$
$$= \frac{1}{m^2} \log\frac{a}{a'} \log\frac{a}{b'} \{\phi(\infty) - \phi(0)\},$$

under the condition $ab = a'b'$, are readily proved, using instead of (1)

$$\int_0^\infty \frac{\phi(ax^m) - \phi(bx^m)}{x}\, dx = \frac{1}{m} \{\phi(\infty) - \phi(0)\} \log\frac{a}{b},$$

and the corresponding formula instead of (2).

NOTE.*—In formula (3), $ab = a'b'$, and factor in the result is $\log\frac{a}{a'} \log\frac{a}{b'}$.

Write $\log a$, $\log b$, $\log a'$, $\log b' = \alpha, \beta, \alpha', \beta'$; then $\alpha + \beta = \alpha' + \beta'$, and $\log\frac{a}{a'} \log\frac{a}{b'} = (\alpha - \alpha')(\alpha - \beta') = \alpha^2 - \alpha(\alpha' + \beta') + \alpha'\beta', = \alpha^2 - \alpha(\alpha + \beta) + \alpha'\beta', = \alpha'\beta' - \alpha\beta$, viz., the factor is $= -\{\log a \log b - \log a' \log b'\}$.

So, in formula (5),

$$abc = a'b'c' \text{ and } \log\frac{a}{a'} \log\frac{a}{b'} \log\frac{a}{c'} = \log\frac{a}{c'} \log\frac{b}{c'} \log\frac{c}{c'}.$$

Factor in result is $\log\frac{a}{a'} \log\frac{a}{b'} \log\frac{a}{c'}$.

Write $\log a$, &c. $= \alpha, \beta, \gamma, \alpha', \beta', \gamma'$; then $\alpha + \beta + \gamma = \alpha' + \beta' + \gamma'$; and it is clear *à priori*, that the factor should be expressible in any one of the six forms

$$\text{factor} = \alpha - \alpha' . \alpha - \beta' . \alpha - \gamma' = \alpha - \alpha' . \beta - \alpha' . \gamma - \alpha'$$
$$= \beta - \alpha' . \beta - \beta' . \beta - \gamma' = \alpha - \beta' . \beta - \beta' . \gamma - \beta'$$
$$= \gamma - \alpha' . \gamma - \beta' . \gamma - \gamma' = \alpha - \gamma' . \beta - \gamma' . \gamma - \gamma'.$$

It at once appears that, besides $\alpha + \beta + \gamma = \alpha' + \beta' + \gamma'$, we must have $\beta\gamma + \gamma\alpha + \alpha\beta = \beta'\gamma' + \gamma'\alpha' + \alpha'\beta'$; viz., these two relations existing, each form of the factor reduces itself to $\alpha\beta\gamma - \alpha'\beta'\gamma'$, thus

$$\alpha - \alpha' . \alpha - \beta' . \alpha - \gamma' = \alpha^3 - \alpha^2 . (\alpha + \beta + \gamma) + \alpha(\beta\gamma + \gamma\alpha + \alpha\beta) - \alpha'\beta'\gamma'$$
$$= \alpha\beta\gamma - \alpha'\beta'\gamma'.$$

Thus relations between the constants are

$$abc = a'b'c', \text{ *i.e.*, } \log a + \log b + \log c = \log a' + \log b' + \log c',$$

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* For this note I have to thank Professor Cayley.

and $\log b \log c + \log c \log a + \log a \log b$

$$= \log b' \log c' + \log c' \log a' + \log a' \log b',$$

and $\text{factor} = \log a \log b \log c - \log a' \log b' \log c'.$

It is presumed that, in the other cases considered, the forms of the factors might be modified in like manner, but this has not been examined.

On certain Identical Differential Relations.

By J. W. L. Glaisher, M.A., F.R.S.

[*Read November 9th*, 1876.]

1. In the "Nouvelle Correspondance Mathématique," Tom. ii., pp. 240—243 (August, 1876), I have shown that the function $e^{\sqrt{x}}$ possesses the curious property that its $(n+1)^{\text{th}}$ differential coefficient is equal to its n^{th} integral, to a power of $4x$ *près*; that is to say, that

$$\left(\int^x dx\right)^n e^{\sqrt{x}} = (4x)^{n+\frac{1}{2}} \left(\frac{d}{dx}\right)^{n+1} e^{\sqrt{x}};$$

or, otherwise, that

$$2^{2n+1} \left(\frac{d}{dx}\right)^n x^{n+\frac{1}{2}} \left(\frac{d}{dx}\right)^{n+1} e^{\sqrt{x}} = e^{\sqrt{x}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

This result is there obtained in the way in which I was led to it, viz., by means of the integral

$$\int_0^\infty e^{-ax^2 - \frac{b}{x^2}}\, dx = \frac{\sqrt{\pi}}{2} \frac{e^{-2\sqrt{(ab)}}}{\sqrt{a}};$$

but it can be proved more easily by expanding $e^{\sqrt{x}}$ in ascending powers of $\sqrt{x}$, that is, by replacing $e^{\sqrt{x}}$ by $1 + x^{\frac{1}{2}} + \frac{x}{1.2} + \frac{x^{\frac{3}{2}}}{1.2.3} + \&c.$, when it is readily seen that this series reproduces itself. It is rather interesting to note how this reproduction is brought about by the differentiations.

2. If in (1) we write $a\sqrt{x}$ for $\sqrt{x}$, we have

$$\left(\frac{d}{dx}\right)^n x^{n+\frac{1}{2}} \left(\frac{d}{dx}\right)^{n+1} e^{a\sqrt{x}} = \frac{a^{2n+1}}{2^{2n+1}} e^{a\sqrt{x}}$$

$$= \frac{1}{2^{2n+1}} \left(\frac{d}{d \,.\, \sqrt{x}}\right)^{2n+1} e^{a\sqrt{x}};$$

whence, if $\phi(\sqrt{x}) = Ae^{a\sqrt{x}} + Be^{b\sqrt{x}} + Ce^{c\sqrt{x}} + \&c.$,

$$\left(\frac{d}{dx}\right)^n x^{n+\frac{1}{2}} \left(\frac{d}{dx}\right)^{n+1} \phi(\sqrt{x}) = \frac{1}{2^{2n+1}} \left(\frac{d}{d \cdot \sqrt{x}}\right)^{2n+1} \phi(\sqrt{x})$$

$$= \frac{1}{2^{2n+1}} \phi^{(2n+1)}(\sqrt{x}) \quad \ldots\ldots\ldots\ldots\ldots(2),$$

where $\phi^{(r)}(u)$ denotes the r^{th} derivative of $\phi(u)$ with regard to u. And since the equation (2) merely expresses an identical relation between the derivatives of $\phi(\sqrt{x})$, we see that it must be true for all forms of ϕ.

3. This relation can, however, be obtained more simply; for

$$\left(\frac{d}{dx}\right)^{n+1} x^p = p\,(p-1) \ldots (p-n)\, x^{p-n-1};$$

therefore $$x^{n+\frac{1}{2}} \left(\frac{d}{dx}\right)^{n+1} x^p = p\,(p-1) \ldots (p-n)\, x^{p-\frac{1}{2}},$$

and

$$\left(\frac{d}{dx}\right)^n x^{n+\frac{1}{2}} \left(\frac{d}{dx}\right)^{n+1} x^p$$

$$= p\,(p-1) \ldots (p-n) \cdot (p-\tfrac{1}{2})\,(p-\tfrac{3}{2}) \ldots (p-n+\tfrac{1}{2})\, x^{p-n-\frac{1}{2}}$$

$$= \frac{2p\,(2p-1)\,(2p-2) \ldots (2p-2n)}{2^{2n+1}} (\sqrt{x})^{2p-2n-1}$$

$$= \frac{1}{2^{2n+1}} \left(\frac{d}{d \cdot \sqrt{x}}\right)^{2n+1} (\sqrt{x})^{2p} = \frac{1}{2^{2n+1}} \left(\frac{d}{d \cdot \sqrt{x}}\right)^{2n+1} x^p,$$

whence $$\left(\frac{d}{dx}\right)^n x^{n+\frac{1}{2}} \left(\frac{d}{dx}\right)^{n+1} \phi(\sqrt{x}) = \frac{1}{2^{2n+1}} \phi^{(2n+1)}(\sqrt{x}) \quad \ldots\ldots\ldots (2).$$

4. As an example of this identity, put $n = 2$: we have

$$\left(\frac{d}{dx}\right)^3 \phi(\sqrt{x}) = \frac{1}{8} \left\{ \frac{1}{x^{\frac{3}{2}}} \phi'''(\sqrt{x}) - \frac{3}{x^2} \phi''(\sqrt{x}) + \frac{3}{x^{\frac{5}{2}}} \phi'(\sqrt{x}) \right\}.$$

Multiply by $x^{\frac{5}{2}}$, and the right-hand side is

$$\tfrac{1}{8} \{ x\phi'''(\sqrt{x}) - 3x^{\frac{1}{2}} \phi''(\sqrt{x}) + 3\phi'(\sqrt{x}) \}.$$

Differentiate, and this becomes

$$\tfrac{1}{16} \{ x^{\frac{1}{2}} \phi^{\text{IV}}(\sqrt{x}) - \phi'''(\sqrt{x}) \},$$

which, after another differentiation, becomes $\frac{1}{32} \phi^{\text{V}}(\sqrt{x})$.

5. The question naturally arises, What are the relations that correspond to (2) when the arguments are $\sqrt[3]{x}$, $\sqrt[4]{x}$, … $\sqrt[r]{x}$? These are found to be

$$\left(\frac{d}{dx}\right)^n x^{n+\frac{2}{3}} \left(\frac{d}{dx}\right)^n x^{n+\frac{1}{3}} \left(\frac{d}{dx}\right)^{n+1} \phi\left(\sqrt[3]{x}\right) = \frac{1}{3^{3n+1}}\, \phi^{(3n+1)}\left(\sqrt[3]{x}\right),$$

$$\left(\frac{d}{dx}\right)^n x^{n+\frac{3}{4}} \left(\frac{d}{dx}\right)^n x^{n+\frac{2}{4}} \left(\frac{d}{dx}\right)^n x^{n+\frac{1}{4}} \left(\frac{d}{dx}\right)^{n+1} \phi\left(\sqrt[4]{x}\right) = \frac{1}{4^{4n+1}}\, \phi^{(4n+1)}\left(\sqrt[4]{x}\right),$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$\left(\frac{d}{dx}\right)^n x^{n+\frac{1}{r}} \left(\frac{d}{dx}\right)^n x^{n+\frac{1}{r}} \ldots \left(\frac{d}{dx}\right)^n x^{n+\frac{1}{r}} \left(\frac{d}{dx}\right)^{n+1} \phi\left(\sqrt[r]{x}\right) = \frac{1}{r^{rn+1}}\, \phi^{(rn+1)}\left(\sqrt[r]{x}\right);$$

and this relation may be written

$$\left\{\left(\frac{d}{dx}\right)^n x^{n+\frac{1}{r}}\right\}^{r-1} \left(\frac{d}{dx}\right)^{n+1} \phi\left(\sqrt[r]{x}\right) = \frac{1}{r^{rn+1}}\, \phi^{(rn+1)}\left(\sqrt[r]{x}\right).$$

The method of demonstration will be evident from an example: thus, to prove the equation in which the argument is $\sqrt[3]{x}$, we have

$$\left(\frac{d}{dx}\right)^{n+1} x^p = p\,(p-1)\,\ldots\,(p-n)\,x^{p-n-1}.$$

Multiply x^{p-n-1} by $x^{n+\frac{2}{3}}$, which gives $x^{p-\frac{2}{3}}$; this differentiated n times becomes $\left(p-\frac{2}{3}\right)\left(p-\frac{5}{3}\right)\ldots\left(p-n+\frac{1}{3}\right)x^{p-n-\frac{2}{3}}$.

Multiply $x^{p-n-\frac{2}{3}}$ by $x^{n+\frac{1}{3}}$, which gives $x^{p-\frac{1}{3}}$; this differentiated n times becomes $\left(p-\frac{1}{3}\right)\left(p-\frac{4}{3}\right)\ldots\left(p-n+\frac{2}{3}\right)x^{p-n-\frac{1}{3}}$;

so that

$$\begin{aligned}
&\left\{\left(\frac{d}{dx}\right)^n x^{n+\frac{1}{3}}\right\}^2 \left(\frac{d}{dx}\right)^{n+1} x^p \\
&\quad = \quad 3p \qquad (3p-3)\,\ldots\,(3p-3n+3)\,(3p-3n) \\
&\qquad \times (3p-2)\,(3p-5)\,\ldots\,(3p-3n+1) \\
&\qquad \times (3p-1)\,(3p-4)\,\ldots\,(3p-3n+2)\,.\,\frac{\left(\sqrt[3]{x}\right)^{3p-3n-1}}{3^{3n+1}} \\
&\quad = \frac{3p\,(3p-1)\,\ldots\,(3p-3n)}{3^{3n+1}}\ \sqrt[3]{x}^{3p-3n-1} \\
&\quad = \frac{1}{3^{3n+1}} \left(\frac{d}{d\,.\,\sqrt[3]{x}}\right)^{3n+1} x^p.
\end{aligned}$$

It is at once evident that the same mode of proof applies to the other equations.

6. The system of equations obtained in the last section may (by replacing $\sqrt{x}$, $\sqrt[3]{x}$, $\sqrt[4]{x}$, ... $\sqrt[r]{x}$ by x in the respective equations) be written in the forms

$$\left(\frac{1}{x}\frac{d}{dx}\right)^n x^{2n+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{n+1}\phi(x)=\phi^{(2n+1)}(x),$$

$$\left(\frac{1}{x^2}\frac{d}{dx}\right)^n x^{3n+1}\left(\frac{1}{x^2}\frac{d}{dx}\right)^n x^{3n+1}\left(\frac{1}{x^2}\frac{d}{dx}\right)^{n+1}\phi(x)=\phi^{(3n+1)}(x),$$

$$\left\{\left(\frac{1}{x^3}\frac{d}{dx}\right)^n x^{4n+1}\right\}^3\left(\frac{1}{x^3}\frac{d}{dx}\right)^{n+1}\phi(x)=\phi^{(4n+1)}(x),$$

$$\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots$$

$$\left\{\left(\frac{1}{x^{r-1}}\frac{d}{dx}\right)^n x^{rn+1}\right\}^{r-1}\left(\frac{1}{x^{r-1}}\frac{d}{dx}\right)^{n+1}\phi(x)=\phi^{(rn+1)}(x).$$

7. These are not the only equations analogous to (2), for it is easy to show, exactly as in § 5, that

$$x^{\frac{1}{3}}\left(\frac{d}{dx}\right)^n x^{n-\frac{1}{3}}\left(\frac{d}{dx}\right)^n x^{n+\frac{2}{3}}\left(\frac{d}{dx}\right)^{n+1}\phi(\sqrt[3]{x})=\frac{1}{3^{3n+1}}\phi^{(3n+1)}(\sqrt[3]{x}),$$

$$x^{\frac{2}{4}}\left(\frac{d}{dx}\right)^n x^{n-\frac{1}{4}}\left(\frac{d}{dx}\right)^n x^{n-\frac{1}{4}}\left(\frac{d}{dx}\right)^n x^{n+\frac{2}{4}}\left(\frac{d}{dx}\right)^{n+1}\phi(\sqrt[4]{x})=\frac{1}{4^{4n+1}}\phi^{(4n+1)}(\sqrt[4]{x}),$$

and generally

$$x^{1-\frac{2}{r}}\left\{\left(\frac{d}{dx}\right)^n x^{n-\frac{1}{r}}\right\}^{r-2}\left(\frac{d}{dx}\right)^n x^{n+1-\frac{1}{r}}\left(\frac{d}{dx}\right)^{n+1}\phi(\sqrt[r]{x})$$
$$=\frac{1}{r^{rn+1}}\phi^{(rn+1)}(\sqrt[r]{x}),$$

which, replacing $\sqrt[r]{x}$ by x, may be written

$$x^{r-2}\left\{\left(\frac{1}{x^{r-1}}\frac{d}{dx}\right)^n x^{rn-1}\right\}^{r-2}\left(\frac{1}{x^{r-1}}\frac{d}{dx}\right)^n x^{rn+r-1}\left(\frac{1}{x^{r-1}}\frac{d}{dx}\right)^{n+1}\phi(x)$$
$$=\phi^{(rn+1)}(x);$$

and of course other systems of equations would be obtained, if the groups of factors that arise in the demonstrations, as in § 5, were formed in different orders. But the system in §§ 5 and 6 is the one in which the law governing the operations is most regular.

8. As a particular case of the results in § 5, we may put $\phi(x)=e^x$, whence

$$\left(\frac{d}{dx}\right)^n x^{n+\frac{1}{3}}\left(\frac{d}{dx}\right)^n x^{n-\frac{1}{3}}\left(\frac{d}{dx}\right)^{n+1}e^{\sqrt[3]{x}}=\frac{1}{3^{3n+1}}e^{\sqrt[3]{x}},$$

and generally

$$\left\{\left(\frac{d}{dx}\right)^n x^{n+\frac{1}{r}}\right\}^{r-1}\left(\frac{d}{dx}\right)^{n+1}e^{\sqrt[r]{x}}=\frac{1}{r^{rn+1}}e^{\sqrt[r]{x}}.$$

9. If the equation (2) be developed, it will be found that it assumes the following form: —

$$u_{2n+1}+\frac{(2n+1)\,2n}{1\,!}\left(\frac{1}{4x}\right)u_{2n}+\frac{(2n+1)\,2n\,(2n-1)\,(2n-2)}{2\,!}\left(\frac{1}{4x}\right)^2 u_{2n-1}$$
$$+\frac{(2n+1)\,2n\,\ldots\,(2n-4)}{3\,!}\left(\frac{1}{4x}\right)^3 u_{2n-2}\,\ldots\,+\frac{(2n+1)\,2n\,\ldots\,4}{(n-1)\,!}\left(\frac{1}{4x}\right)^{n-1} u_{n+2}$$
$$+\frac{(2n+1)\,!}{n\,!}\left(\frac{1}{4x}\right)^n u_{n+1}=\left(\frac{1}{4x}\right)^{n+\frac{1}{2}}\phi^{(2n+1)}(\sqrt{x}),$$

where u_r is written for $\left(\frac{d}{dx}\right)^r\phi(\sqrt{x})$.

10. Professor Cayley has remarked to me that the general property upon which the foregoing transformations depend is as follows:—The symbols $x^{1-\alpha}\frac{d}{dx}x^{\alpha}$, $x^{1-\beta}\frac{d}{dx}x^{\beta}$, &c. are convertible; so that, operating with a set of such symbols upon a function of x, the result is the same no matter in what order the operations are performed. And, to prove the convertibility, it is only necessary to remark that we have

$$x^{1-\alpha}\frac{d}{dx}x^{\alpha}.\,x^m=(m+\alpha)\,x^m,$$

viz, the operation performed upon the arbitrary power x^m of x is equivalent to a multiplication by $m+\alpha$; hence the symbols $x^{1-\alpha}\frac{d}{dx}x^{\alpha}$ and $x^{1-\beta}\frac{d}{dx}x^{\beta}$ are convertible in regard to the operand x^m; and by considering any given function of x as a sum of terms Ax^m, it thence appears that they are convertible when the operand is ϕx, a given function of x; but it is also easy to show directly that they are convertible in regard to the operand ϕx.

Thus the equation (2), putting therein x^2 for x, becomes

$$\left(x^{-1}\frac{d}{dx}\right)^n x^{2n+1}\left(x^{-1}\frac{d}{dx}\right)^{n+1}\phi x=\left(\frac{d}{dx}\right)^{2n+1}\phi x.$$

If, to fix the ideas, $n=5$, here writing $x\phi x$ instead of ϕx, the general theorem gives

$$x^5\frac{d}{dx}x^{-4}.\,x^3\frac{d}{dx}x^{-2}.\,x^6\frac{d}{dx}x^{-5}.\,x^4\frac{d}{dx}x^{-3}.\,x^2\frac{d}{dx}x^{-1}.\,x\phi x$$
$$=x^6\frac{d}{dx}x^{-5}.\,x^5\frac{d}{dx}x^{-4}.\,x^4\frac{d}{dx}x^{-3}.\,x^3\frac{d}{dx}x^{-2}.\,x^2\frac{d}{dx}x^{-1}.\,x\phi x.$$

The right-hand side is here $=x^6\left(\frac{d}{dx}\right)^5\phi x$; the left-hand side $=x^5\frac{d}{dx}x^{-1}\frac{d}{dx}x^4\frac{d}{dx}x^{-1}\frac{d}{dx}x^{-1}\frac{d}{dx}\phi x$, that is, $x^6\left(x^{-1}\frac{d}{dx}\right)^2 x^5\left(x^{-1}\frac{d}{dx}\right)^3\phi x$; or, omitting the common factor x^6, the equation is

$$\left(x^{-1}\frac{d}{dx}\right)^2 x^5\left(x^{-1}\frac{d}{dx}\right)^3\phi x=\left(\frac{d}{dx}\right)^5\phi x,$$

which is right.

December 14*th*, 1876.

LORD RAYLEIGH, F.R.S., President, in the Chair.

Mr. R. F. Davis and Mr. H. Weston Eve were elected m mbers, and Mr. G. W. Von Tunzelmann was proposed for election.

Prof. Greenhill (the Auditor) reported that he had audited the accounts of the Society, comparing them with the vouchers, and found them correct. On the motion of Mr. Merrifield, seconded by Prof. Clifford, a vote of thanks was accorded to the Auditor.

The following papers were read or communicated :—

"On the Conditions of Perpendicularity in a Parallelepipedal System," by Prof. H. J. S. Smith, F.R.S., Vice-President.

"On the Condition for the Existence of a Surface cutting at Right Angles a given Set of Lines," by Prof. Cayley, F.R.S. (read by Mr. Glaisher).

"The Orthogonal Transformation," and additions to former paper on "Transformation of Elliptic Functions," by Prof. Clifford, F.R.S.

"On the Simplest Continuous Manifoldness of two Dimensions and of Finite Extent," by Mr. F. W. Frankland (communicated by Mr. Spottiswoode, F.R.S., read by Mr. Tucker).

"On the Theory of Electric Images, and its application to the case of two Charged Spherical Conductors placed opposite one another," by Mr. W. D. Niven, M.A.

"On Viscous Fluids," and "Quaternion Forms of some general Propositions in Fluid Motion," by Mr. J. G. Butcher, B.A., Fellow of Trinity College, Cambridge (communicated by Mr. G. H. Darwin, read in part by Mr. Tucker).

"An easy Method of finding the Invariant Equation expressing any Poristic Relation between Two Conics," by Prof. Wolstenholme (read in part by Mr. Tucker).

The following presents were made to the Society :—

"Proceedings of Royal Society," No. 175.

"Mémoires de la Société des Sciences Physiques . . . de Bordeaux," 2^{e} série, Tome 1er, Paris, 1876.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome onzième, Sept., Oct., 1876.

"Interpolation and Adjustment of Series," by E. L. de Forest, Newhaven, 1876.

"Lessons introductory to the Higher Algebra," by G. Salmon, D.D. Third edition. Dublin, 1876. From the Author.

"Journal of Education," Dec. 1876.

"Weitere Untersuchungen über das Ikosaeder" (aus den Sitzungsberichten der physikalisch-medicinischen Societät zu Erlangen, 13th Nov. 1876), von F. Klein.

On the Condition for the Existence of a Surface cutting at Right Angles a given Set of Lines. By Prof. Cayley.

[*Read December 14th*, 1876.]

In a congruency or doubly infinite system of right lines, the direction-cosines α, β, γ of the line through any given point (x, y, z), are expressible as functions of x, y, z; and it was shown by Sir W. R. Hamilton in a very elegant manner that, in order to the existence of a surface (or, what is the same thing, a set of parallel surfaces) cutting the lines at right angles, $\alpha dx + \beta dy + \gamma dz$ must be an exact differential: when this is so, writing $V = \int(\alpha dx + \beta dy + \gamma dz)$, we have $V = c$, the equation of the system of parallel surfaces each cutting the given lines at right angles.

The proof is as follows:—If the surface exists, its differential equation is $\alpha dx + \beta dy + \gamma dz = 0$, and this equation must therefore be integrable by a factor. Now the functions α, β, γ are such that $\alpha^2 + \beta^2 + \gamma^2 = 1$, and they besides satisfy a system of partial differential equations which Hamilton deduces from the geometrical notion of a congruency; viz., passing from the point (x, y, z) to the consecutive point on the line, that is, the point whose coordinates are $x + \rho\alpha$, $y + \rho\beta$, $z + \rho\gamma$ (ρ infinitesimal), the line belonging to this point is the original line; and consequently α, β, γ, considered as functions of x, y, z, must remain unaltered when these variables are changed into $x + \rho\alpha, y + \rho\beta, z + \rho\gamma$, respectively. We thus obtain the equations

$$\alpha\frac{d\alpha}{dx} + \beta\frac{d\alpha}{dy} + \gamma\frac{d\alpha}{dz} = 0,$$

$$\alpha\frac{d\beta}{dx} + \beta\frac{d\beta}{dy} + \gamma\frac{d\beta}{dz} = 0,$$

$$\alpha\frac{d\gamma}{dx} + \beta\frac{d\gamma}{dy} + \gamma\frac{d\gamma}{dz} = 0.$$

Combining herewith the equations obtained by differentiation of $\alpha^2 + \beta^2 + \gamma^2 = 1$, viz.,

$$\alpha\frac{d\alpha}{dx} + \beta\frac{d\beta}{dx} + \gamma\frac{d\gamma}{dx} = 0,$$

$$\alpha\frac{d\alpha}{dy} + \beta\frac{d\beta}{dy} + \gamma\frac{d\gamma}{dy} = 0,$$

$$\alpha\frac{d\alpha}{dz} + \beta\frac{d\beta}{dz} + \gamma\frac{d\gamma}{dz} = 0,$$

and subtracting the corresponding equations, we obtain three equations which may be written

$$\alpha : \beta : \gamma = \frac{d\beta}{dz} - \frac{d\gamma}{dy} : \frac{d\gamma}{dx} - \frac{d\alpha}{dz} : \frac{d\alpha}{dy} - \frac{d\beta}{dx},$$

or, what is the same thing,

$$\frac{d\beta}{dz}-\frac{d\gamma}{dy},\quad \frac{d\gamma}{dx}-\frac{d\alpha}{dz},\quad \frac{d\alpha}{dy}-\frac{d\beta}{dx}=k\alpha,\ k\beta,\ k\gamma,$$

and, multiplying by α, β, γ, and adding,

$$k=\alpha\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)+\beta\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right)+\gamma\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right).$$

We thus see that, if the function on the right-hand vanish, then $k=0$, and consequently also $\frac{d\beta}{dz}-\frac{d\gamma}{dy}$, $\frac{d\gamma}{dx}-\frac{d\alpha}{dz}$, $\frac{d\alpha}{dy}-\frac{d\beta}{dx}$ each $=0$; viz., if the equation $\alpha dx+\beta dy+\gamma dz=0$ be integrable, then $\alpha dx+\beta dy+\gamma dz$ is an exact differential; which is the theorem in question.

But it is interesting to obtain the first mentioned set of differential equations from the analytical equations of a congruency, viz., these are $x=mz+p$, $y=nz+q$, where m, n, p, q are functions of two arbitrary parameters, or, what is the same thing, p, q are given functions of m, n; and therefore, from the three equations, m, n are given functions of x, y, z. And it is also interesting to express in terms of these quantities m, n, considered as functions of x, y, z, the condition for the existence of the set of surfaces.

We have

$$\alpha,\beta,\gamma=\frac{m}{\mathrm{R}},\ \frac{n}{\mathrm{R}},\ \frac{1}{\mathrm{R}},\quad \text{where } \mathrm{R}=\sqrt{1+m^2+n^2};$$

and thence without difficulty

$$\left(\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz}\right)\alpha$$
$$=\frac{1}{\mathrm{R}^4}\left[(1+n^2)\left(m\frac{dm}{dx}+n\frac{dm}{dy}+\frac{dm}{dz}\right)-mn\left(m\frac{dn}{dx}+n\frac{dn}{dy}+\frac{dn}{dz}\right)\right],$$

$$\left(\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz}\right)\beta$$
$$=\frac{1}{\mathrm{R}^4}\left[-mn\left(\quad,,\quad\right)+(1+m^2)\left(\quad,,\quad\right)\right],$$

$$\left(\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz}\right)\gamma$$
$$=\frac{1}{\mathrm{R}^4}\left[-\left(\quad,,\quad\right)-\left(\quad,,\quad\right)\right],$$

so that the required equations in α, β, γ will be satisfied if only

$$m\frac{dm}{dx}+n\frac{dm}{dy}+\frac{dm}{dz}=0,$$
$$m\frac{dn}{dx}+n\frac{dn}{dy}+\frac{dn}{dz}=0,$$

and it is to be shown that these equations hold good.

Writing for shortness $dp = A dm + B dn$, $dq = C dm + D dn$, the equations of the line give

$$\left.\begin{aligned} 1 &= z\frac{dm}{dx} + A\frac{dm}{dx} + B\frac{dn}{dx}, \\ 0 &= z\frac{dm}{dy} + A\frac{dm}{dy} + B\frac{dn}{dy}, \\ -m &= z\frac{dm}{dz} + A\frac{dm}{dz} + B\frac{dn}{dz}, \end{aligned}\right| \quad \left.\begin{aligned} 0 &= z\frac{dn}{dx} + C\frac{dm}{dx} + D\frac{dn}{dx}, \\ 1 &= z\frac{dn}{dy} + C\frac{dm}{dy} + D\frac{dn}{dy}, \\ -n &= z\frac{dn}{dz} + C\frac{dm}{dz} + D\frac{dn}{dz}, \end{aligned}\right.$$

or, writing $\lambda, \mu, \nu = \frac{dm}{dy}\frac{dn}{dz} - \frac{dm}{dz}\frac{dn}{dy}, \ \frac{dm}{dz}\frac{dn}{dx} - \frac{dm}{dx}\frac{dn}{dz}, \ \frac{dm}{dx}\frac{dn}{dy} - \frac{dm}{dy}\frac{dn}{dx}$, so that identically

$$\lambda\frac{dm}{dx} + \mu\frac{dm}{dy} + \nu\frac{dm}{dz} = 0,$$
$$\lambda\frac{dn}{dx} + \mu\frac{dn}{dy} + \nu\frac{dn}{dz} = 0;$$

then in each set multiplying by λ, μ, ν and adding, so as to eliminate A, B, C, D, we find

$$\lambda - m\nu = 0, \qquad \mu - n\nu = 0,$$

and substituting these values of λ, μ in the last preceding equations, ν divides out, and we have the two equations in question.

The foregoing equations give further

$$A, B, C, D = -z + \frac{1}{\nu}\frac{dn}{dy}, \quad -\frac{1}{\nu}\frac{dm}{dy}, \quad -\frac{1}{\nu}\frac{dn}{dx}, \quad -z + \frac{1}{\nu}\frac{dm}{dx}.$$

Taking for α, β, γ the before-mentioned values, we find

$$\frac{d\alpha}{dy} - \frac{d\beta}{dx} = \frac{1}{R}\left(\frac{dm}{dy} - \frac{dn}{dx}\right) - \frac{m}{R^3}\left(m\frac{dm}{dy} + n\frac{dn}{dy}\right) - \frac{n}{R^3}\left(m\frac{dm}{dx} + n\frac{dn}{dx}\right)$$
$$= \frac{1}{R^3}\left\{(1+n^2)\frac{dm}{dy} - (1+m^2)\frac{dn}{dx} + mn\left(\frac{dm}{dx} - \frac{dn}{dy}\right)\right\};$$

and similarly, but using the equations

$$m\frac{dm}{dx} + n\frac{dm}{dy} + \frac{dm}{dz} = 0, \qquad m\frac{dn}{dx} + n\frac{dn}{dy} + \frac{dn}{dz} = 0,$$

to eliminate the coefficients $\frac{dm}{dz}$, $\frac{dn}{dz}$ which in the first instance present themselves, we find

$$\frac{d\beta}{dz} - \frac{d\gamma}{dy} = \frac{m}{R^3}\left\{ \quad ,, \quad ,, \quad ,, \quad \right\},$$
$$\frac{d\gamma}{dx} - \frac{d\alpha}{dz} = \frac{n}{R^3}\left\{ \quad ,, \quad ,, \quad ,, \quad \right\},$$

whence, multiplying by γ, α, β, and adding,

$$\alpha\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)+\beta\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right)+\gamma\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right)$$
$$=\frac{1}{1+m^2+n^2}\left\{(1+n^2)\frac{dm}{dy}-(1+m^2)\frac{dn}{dx}+mn\left(\frac{dm}{dx}-\frac{dn}{dy}\right)\right\},$$

or we have

$$(1+n^2)\frac{dm}{dy}-(1+m^2)\frac{dn}{dx}+mn\left(\frac{dm}{dx}-\frac{dn}{dy}\right)=0$$

as the condition for the existence of the set of surfaces.

It is clear that the condition is satisfied when the lines are the normals of a given surface: seeking the surfaces which cut the lines at right angles, we obtain the parallel surfaces; and we are led to the theorem that any parallel surface is the locus of the extremity of a line of constant length measured off from each point of the surface along the normal—or, what is equivalent thereto, the parallel surface is the envelope of a sphere of constant radius having its centre on the surface. I will verify the theorem for the case of the ellipsoid. Taking X, Y, Z as the coordinates of a point on the ellipsoid $\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}=1$, and x, y, z as current coordinates, the equations of the normal are

$$\frac{a^2}{X}(x-X)=\frac{b^2}{Y}(y-Y)=\frac{c^2}{Z}(z-Z)\ (=\lambda \text{ suppose}).$$

We have therefore

$$X, Y, Z=\frac{a^2x}{a^2+\lambda},\quad \frac{b^2y}{b^2+\lambda},\quad \frac{c^2z}{c^2+\lambda},$$

and thence

$$\frac{a^2x^2}{(a^2+\lambda)^2}+\frac{b^2y^2}{(b^2+\lambda)^2}+\frac{c^2z^2}{(c^2+\lambda)^2}=1,$$

an equation which determines λ as a function of x, y, z.

The direction-cosines α, β, γ of the normal are proportional to $\frac{X}{a^2}$, $\frac{Y}{b^2}$, $\frac{Z}{c^2}$, that is, to $\frac{x}{a^2+\lambda}$, $\frac{y}{b^2+\lambda}$, $\frac{z}{c^2+\lambda}$, and the equation $\alpha^2+\beta^2+\gamma^2=1$ then determines their absolute magnitudes: the equation $\alpha dx+\beta dy+\gamma dz=dV$ thus is

$$\frac{\frac{x\,dx}{a^2+\lambda}+\frac{y\,dy}{b^2+\lambda}+\frac{z\,dz}{c^2+\lambda}}{\sqrt{\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2}}}=dV,$$

viz., the left-hand side, considering therein λ as a given function of V, is an exact differential. We verify this by finding the value of V, viz.,

writing down the two equations

$$\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2}-\frac{\mathrm{V}^2}{\lambda^2}=0,$$

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}-\frac{\mathrm{V}^2}{\lambda}=1,$$

these are equivalent in virtue of the equation that determines λ; and it is to be shown that, regarding V as given by either of them, say by the second equation, we have for $d\mathrm{V}$ its foregoing value. In fact, differentiating the second equation, the term in $d\lambda$ disappears by virtue of the first equation, and the result is

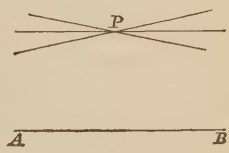
$$\frac{x\,dx}{a^2+\lambda}+\frac{y\,dy}{b^2+\lambda}+\frac{z\,dz}{c^2+\lambda}-\frac{\mathrm{V}d\mathrm{V}}{\lambda}=0,$$

in which substituting for $\frac{\mathrm{V}}{\lambda}$ its value from the first equation, we have for $d\mathrm{V}$ the value in question. Regarding V as a given constant, the two equations give, by elimination of λ, an equation $\varphi(x, y, z, \mathrm{V})=0$, which is in fact the surface parallel to the ellipsoid and at a constant normal distance $=\mathrm{V}$ from it.

On the Simplest Continuous Manifoldness of two Dimensions and of Finite Extent. By F. W. Frankland, Esq.

[*Read December 14th*, 1876.]

One of the most remarkable speculations of the present century is the speculation that the axioms of geometry may be only approximately true, and that the actual properties of space may be somewhat different from those which we are in the habit of ascribing to it. It was Lobatchewsky who first worked out the conception of a space in which some of the ordinary laws of geometry should no longer hold good. Among the axioms which lie at the foundation of the Euclidian scheme, he assumed all to be true except the one which relates to parallel straight lines. An equivalent form of this axiom, and the one now generally employed in works on geometry, is the statement that it is impossible to draw more than one straight line parallel to a given straight line through a given point outside it. In other words, if we take a fixed straight line AB prolonged infinitely in both directions, and a fixed point P outside it; then, if a second straight line, also infinitely prolonged in both directions, be made to rotate about P, there is



only one position in which it will not intersect AB. Now Lobatchewsky made the supposition that this axiom should be untrue, and that there should be a finite angle through which the rotating line might be turned without ever intersecting the fixed straight line AB. And in following out the consequences of this assumption he was never brought into collision with any of the other axioms, but was able to construct a perfectly self-consistent scheme of propositions, all of them valid as analytical conceptions, but all of them perfectly incapable of being realised in thought.

Many of the results he arrived at were very curious; such as, for instance, that the three angles of a triangle would not be together equal to two right angles, but would be together less than two right angles by a quantity proportional to the area of the triangle. If we were to increase the sides of such a triangle, keeping them always in the same proportion, the angles would become continually smaller and smaller, until at last the three sides would cease to form a triangle, because they would never meet at all.

There are many other assumptions, at variance with the axioms of Euclid, which may be made respecting distance-relations, and which yield self-consistent schemes of propositions differing widely from the propositions of geometry. We see, therefore, that geometry is only a particular branch of a more general science, and that the conception of space is a particular variety of a wider and more general conception. This wider conception, of which time and space are particular varieties, it has been proposed to denote by the term *manifoldness*. Whenever a general notion is susceptible of a variety of specialisations, the aggregate of all such specialisations is called a manifoldness. Thus space is the aggregate of all *points*, and each point is a specialisation of the general notion of *position*. In the same way time is the aggregate of all *instants*, and each instant is a specialisation of the general notion of *position in time*. Space and time are, in fact, of all manifoldnesses, the ones with which we are by far the most frequently concerned.

Now there is an important feature in which these two manifoldnesses agree. They are both of them of such a nature that no limit can be conceived of their divisibility. However near together two points in space may be, we can always conceive the existence of intermediate points. And the same thing holds in regard to time. Mathematicians express this fact by saying that space and time are *continuous* manifoldnesses. But there is another feature, equally important with the foregoing, in regard to which time and space are strikingly contrasted. If we wish to travel away from any particular instant in time, there are only two directions in which we can set out. We must either ascend or descend the stream. But from a point in space we can set out in an infinite number of directions. This difference is

expressed by saying that time is a manifoldness of *one dimension*, and that space is a manifoldness of *more than one dimension*. An aggregate of points in which we could only travel backwards or forwards would be, *not* solid space, but a *line*. A line, therefore, is a manifoldness of one dimension. A *surface*, again, may be regarded as an aggregate of lines; and it is an aggregate of such a nature, that if we wish to travel away from a particular line, there are only two directions in which we can set out. It is therefore a line-aggregate of one dimension. Considered as a point-aggregate, it has two dimensions, and accordingly it is a manifoldness of two dimensions. In the same way it will be seen that solid space is a manifoldness of three dimensions.

I have endeavoured by these remarks to explain what is meant when we speak of a continuous manifoldness of two dimensions. It is the object of this paper to communicate some results I have arrived at respecting the properties of the simplest of such manifoldnesses which has a finite extent. The existence of the particular manifoldness I shall endeavour to describe has been referred to in a remarkable lecture by Prof. Clifford on "The Postulates of the Science of Space," but, so far as I am aware, its properties have not hitherto been worked out in detail.

The simplest of all doubly-extended continuous manifoldnesses is the *plane*. But it is not a manifoldness of finite extent. It reaches to infinity in every direction, and its area is greater than any assignable area. It is therefore not the manifoldness of which we are in search. Now the circumstance in which the plane differs from those doubly-extended manifoldnesses which are next to it in order of simplicity, is the possibility that figures constructed in it may be magnified or diminished to any extent without alteration of shape; in other words, that figures which can be constructed in it at all can be constructed to any scale. That this property is not possessed by curved surfaces, may be seen by considering the case of a spherical triangle. If the sides of a triangle constructed on a given sphere be all of them increased or diminished in the same proportion, the shape of the triangle will not remain the same. Now it has been found by Prof. Riemann that this property of the plane is equivalent to the following two axioms:—(1) That two geodesic lines which diverge from a point will never intersect again, or, as Euclid puts it, that two straight lines cannot inclose a space; and (2) that two geodesic lines which do not intersect will make equal angles with every other geodesic line. The second is precisely equivalent to Euclid's twelfth axiom. Deny the first of these axioms, and you have a manifoldness of uniform positive curvature; deny the second, and you have one of uniform negative curvature. The plane lies midway between the two, and its curvature is zero at every point.

Let us consider, then, the case of a doubly-extended manifoldness, of which the curvature is uniform and positive. The first of the before-mentioned two axioms is no longer true. Geodesic lines diverging from a point do not continue to diverge for ever. They meet again and inclose a space. The first question which presents itself is with reference to the situation of the point towards which they ultimately converge. In the case of a spherical surface they will converge towards a point which is separated from the starting-point by half the length of a geodesic line. And this is the only case we are able to conceive. The surface of a sphere is the only doubly-extended manifoldness of uniform positive curvature which geometry recognises, and it is the only one which we can figure to ourselves in thought. It is not, however, the *simplest* of such manifoldnesses. To obtain the simplest case we must suppose that the point towards which two geodesic lines converge is separated from their starting-point, not by *half*, but by the *entire* length of a geodesic line; or, what amounts to the same thing, that it *coincides* with the starting-point. It is true that we are utterly unable to figure to ourselves a surface in which two geodesic lines shall have only one point of intersection, and shall yet inclose a space. But we are perfectly at liberty to reason about such a surface, because there is nothing self-contradictory in the definition of it, and because, therefore, the analytical conception of it is perfectly valid. It is the simplest continuous manifoldness of two dimensions, and of finite extent, and those few properties of it which I have worked out appear to me to be very beautiful. In order to make my observations more intelligible, I shall for the future speak of it as a surface, and its geodesic lines I shall speak of as straight lines. I have the highest authority for using this nomenclature, and though it will impart to my theorems a very paradoxical sound, it is calculated, I think, to give a juster idea of their meaning than if I were to use the more accurate but less familiar terms.

Assuming, then, as the fundamental properties of our surface, that every straight line is of finite extent (in other words, that a point moving along it will arrive at the position from which it started after travelling a finite distance), and that two straight lines cannot have two points in common, the first corollary I propose to establish is, that all straight lines in the surface are of equal extent.

Let A, B, be two straight lines in the given surface. If possible, let A be greater than B. From A cut off a portion equal to B. Let P, Q, be the extreme points of this segment, and let R be any point in B. Apply the line A to the line B in such a manner that the point P falls on the point R, then, since in a surface of uniform curvature equal lengths of geodesic lines may be made to coincide, the segment PQ will coincide with the entire straight line B. Hence Q will fall upon

R. But P coincides with R, and P and Q do not coincide with one another, since PQ is less than the entire straight line A; therefore Q cannot coincide with R. Hence A cannot be greater than B.

The straight lines here spoken of are, of course, not *terminated* straight lines. What the proposition asserts is, that the *entire* length of all straight lines in the given surface is the same. The corresponding proposition in spherical geometry is, that all great circles of a given sphere are equal.

There are a great many other analogies between the imaginary surface here treated of and the surface of a sphere. Its straight lines, though they are like the straight lines of a plane in the circumstance that any two of them have only one point of intersection, are in many other respects analogous to great circles. In any of its straight lines, for instance, each point has a corresponding point which is *opposite* to it, and farther from it than any other point in the line. For, if by setting out from a point, and travelling a finite distance in a particular direction, we get back to the starting-point, there must be a point halfway on our journey which is farther from the starting-point than any other point in the line, and which may very appropriately be called its opposite point. It is an obvious corollary that the distance between any two points will be the same as the distance between their opposite points.

Let us now consider the case of a number of straight lines radiating from a centre. In each of them there will be a point which is opposite to that centre. And it will be a separate point for every separate straight line. For no two straight lines can have two points in common, and since these radiating lines have a common centre of radiation, they can have no other point in common. Hence, if we suppose one of these lines to rotate about the centre, the point opposite to the centre will describe a continuous line, and one which finally returns into itself. It is the locus of all points in the surface opposite to the centre of radiation. What now is the character of this locus? In the first place, it is a line which is of the same shape all along, and of which all equal segments therefore can be made to coincide. For any two positions of the rotating line which contain a given angle may be placed upon any other two positions which contain an equal angle. Then, since the length of all straight lines in the surface is the same, the opposite points will coincide, and by parity of reasoning all intermediate points of the locus. But in the second place, the locus is also of the same shape on both sides. For each point in it may be approached from the centre of radiation in two different ways, and it is at the same distance from that centre, whether it be approached in the one way or the other. Any particular segment, in fact, of the locus has its extreme points joined to the centre of radiation by lines which

are of equal length, and which include an equal angle—lines, therefore, which may be made to coincide. Since this is the case for any segment whatever, and for every subdivision of a segment, all the points of a segment will still remain on it, if the segment be turned round and applied to itself. Hence the locus is of the same shape, whether viewed from the one side or from the other. But since it is also of the same shape all along, it satisfies Leibnitz's definition of a straight line, and it is, in fact, a geodesic line of the surface.

Hence we have this second proposition—that all points in the surface opposite to a given point lie in a straight line.

From the method of its construction, this straight line is farther from the given point than any other line in the surface. Travelling from the given point as a centre, in whatever direction we might set out, we should, after completing half our journey, arrive at this farthest straight line, we should cross it at right angles, and we should then keep getting nearer and nearer to our starting-point, until we finally reached it from the opposite side.

Each separate point in the surface, moreover, has a separate farthest line. For, if any two points be taken, the points opposite to them on the straight line which joins them will be distinct. Hence their farthest lines will cut this joining line in two separate points. They must, therefore, be two *separate* lines, for the same straight line cannot cut another straight line in two separate points. In a similar manner it may be shown that each straight line in the surface has a separate farthest point. Hence there exists a reciprocal relation between the points and straight lines of the surface, a relation which we may express by saying that every point in the surface has a *polar*, and that every straight line in the surface has a *pole*. It is then easy to show that when a point is made to move along a straight line its polar will turn about a point, and that when a straight line is made to turn about a point its pole will move along a straight line.

It is interesting to compare these propositions with the corresponding ones in spherical geometry. There, too, each point has a farthest geodesic line; that is to say, a geodesic line which is farther from it than any other geodesic line on the sphere. But each geodesic line has *two* farthest points or poles, instead of having only one. Hence there is not that perfect reciprocity of relationship between points and geodesic lines which exists in the surface we have been examining; and this is one of the many ways in which the sphere shows itself to be inferior to that surface in simplicity.

The most astounding fact I have elicited in connection with this surface is one which comes out in the theory of the circle. Defining a circle as the locus of points equidistant from a given point, we shall find that it assumes a very extraordinary shape when its radius is at all

nearly equal to half the entire length of a straight line. For let us again figure to ourselves a number of straight lines radiating from a point. Let l be the total length of each straight line. Then the supposition we have to make is that the radius of our circle shall be nearly equal to $\frac{l}{2}$. Let us suppose it equal to $\frac{l}{2} - m$, where m is small as compared with l. Each of the radiating lines will cut the circle in two points, and each of these points will be at a distance from O equal to $\frac{l}{2} - m$ or $\frac{l}{2} + m$ according as the distance is measured in the one direction or the other. And their distance from each other will be equal to $2m$, that is to say, it will be comparatively small. But each point on the polar of O will be at a distance from O equal to $\frac{l}{2}$. Hence each point on the circle will be at a distance from this polar equal to m. Moreover, every point at a distance of m from the polar will be a point on the circle, because it will be at a distance of $\frac{l}{2} - m$ from O. But the locus of points at a distance of m from the straight line AB will consist of two branches, CD and EF, one on either side of AB, and at the same distance from it along their whole length. It is true that these branches form in reality a single continuous line. A point travelling along from C to D, and further in the same direction, would ultimately appear at E, travel along to F, and then, after a further journey, reappear at the point C. But this does not alter the fact that, when a small portion only of this line is contemplated, it presents the appearance of two straight lines, each of them parallel to, and equidistant from, AB.

In the limiting case, where the radius becomes equal to $\frac{l}{2}$, CD and EF both of them coincide with AB. The circle merges into a straight line, and becomes, in fact, the polar of its own centre. It is not, indeed, quite accurate to say that it merges into a straight line, for it reduces itself rather to two coincident straight lines, and its equation in co-ordinate geometry would be one of the second degree.

In regard to the surface here treated of, it is easy to see that, as with the sphere, the smaller the portion of it we bring under our consideration, the more nearly its properties approach to those of the plane. Indeed, if we consider an area that is very small as compared with the total area of the surface, its properties will not differ sensibly from those of the plane. And on this ground it has been argued that the universe may in reality be of finite extent, and that each of its geodesic lines

may return into itself, provided only that its total magnitude be very great as compared with any magnitude which we can bring under our observation.

In conclusion, I cannot do better than quote the passage in which Prof. Clifford explains what must be the constitution of space if this hypothesis should be true. "In this case," he says, "the universe, as known, is again a valid conception, for the extent of space is a finite number of cubic miles. And this comes about in a curious way. If you were to start in any direction whatever, and move in that direction in a perfect straight line according to the definition of Leibnitz, after travelling a most prodigious distance, to which the parallactic unit—200,000 times the diameter of the earth's orbit—would be only a few steps, you would arrive at—this place. Only, if you had started upwards, you would appear from below. Now, one of two things would be true. Either, when you had got half-way on your journey, you came to a place that is opposite to this, and which you must have gone through, whatever direction you started in, or else all paths you could have taken diverge entirely from each other till they meet again at this place. In the former case every two straight lines in a plane meet in two points, in the latter they meet only in one. Upon this supposition of a positive curvature the whole of geometry is far more complete and interesting; the principle of duality, instead of half breaking down over metric relations. applies to all propositions without exception. In fact, I do not mind confessing that I personally have often found relief from the dreary infinities of homaloidal space in the consoling hope that, after all, this other may be the true state of things."

On the Theory of Electric Images, and its application to the case of two Charged Spherical Conductors. *By* W. D. Niven, Fellow of Trinity College, Cambridge.

[*Read December* 14*th*, 1876.]

The problem of finding the distributions of electricity on two spheres not intersecting has been already solved by several mathematicians. In the first instance by Poisson, in the "Mémoires de l'Institut" for the year 1811, the solution being contained in two papers. In the first of these he supposes the density to be expressed for each sphere in a series of zonal harmonics whose common axis is the line of centres; he then finds in corresponding series the potentials due to the two distributions at points outside and inside the two spheres; and from the conditions that the spheres are at the same potentials at all

internal points, he is led to two functional equations the solution of which would completely determine the electrifications as well as the potentials and charges. The functional equations may be solved in series, but Poisson is not satisfied till he has obtained results expressed in definite integrals. In his first memoir he is able to do this only in the case when the two spheres are in contact. He returns, however, to the general case in the second memoir, and, after obtaining the solution in series, he succeeds in summing the series in the form of definite integrals.

Sir W. Thomson has at different times written on this subject; but his paper "On the mutual Attraction or Repulsion between two Electrified Spherical Conductors," forming one of the reprint of his papers on electrostatics and magnetism, may be taken as a summary of the most important of his results. He supposes uniform distributions on each of the spheres, equivalent in their action on external points to electrified points at the centres, to be reflected backwards and forwards, changing sign each time at each sphere alternately, thereby forming with the two original points four infinite series of electrified points. The resultant potential at any external point due to all these is the required potential, and the charge of each conductor is the sum of the electric points inside of it. The paper is valuable, not only on account of this very interesting application of the theory of images, but also because it contains an important proposition to the effect that the potential energy of a system of conductors is half the sum of the products obtained by multiplying the charge on each conductor into its potential. By means of this theorem the force between the spheres is easily found when the general problem of finding the potentials and the charges has been solved.

A third solution is given by Professor Clerk-Maxwell, and is to be found in his work on "Electricity and Magnetism," pp. 212—221. The successive images in two uninsulated concentric spherical conductors due to an electrified point placed between them are first found, and the system is then inverted with regard to the point. The four systems of images will then invert into four other systems equivalent to those in Sir W. Thomson's solution. In this solution the author uses dipolar coordinates, somewhat simplifying thereby the calculations.

In the solution now given I have employed the method of electric images, regarded however from the analytical point of view, as an interpretation of a secondary solution of La Place's equation. The results arrived at are more allied in form with those of Poisson's solution than with those of the two last mentioned. I have accordingly thought proper to introduce into this paper a brief sketch of the method of that part of Poisson's work which enables

us to express the results in terms of definite integrals. In some cases this can be done here with greater simplicity than in Poisson's memoirs, because the method of inversion enables us to deal with simpler series than those found by that writer.

1. *Preliminary proposition.*—If $f(V, x, y, z) = 0$ be any solution of La Place's equation, then there is also another solution of the form

$$f\left(\frac{r}{a}V,\ \frac{a^2}{r^2}x,\ \frac{a^2}{r^2}y,\ \frac{a^2}{r^2}z\right) = 0,$$

where $r^2 = x^2+y^2+z^2$. For if we put

$$\left.\begin{aligned}\xi &= \frac{a^2}{r^2}x\\ \eta &= \frac{a^2}{r^2}y\\ \zeta &= \frac{a^2}{r^2}z\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and change the independent variables from x, y, z to ξ, η, ζ, or use the general transformation of La Place's equation, treating equations (1) as three surfaces cutting orthogonally, ξ, η, ζ being the variable parameters, we easily obtain

$$\frac{d\mathrm{V}}{dx^2}+\frac{d^2\mathrm{V}}{dy^2}+\frac{d^2\mathrm{V}}{dz^2} = \frac{a^5}{r^5}\left\{\frac{d^2\left(\frac{r\mathrm{V}}{a}\right)}{d\xi^2}+\frac{d^2\left(\frac{\mathrm{V}}{a}\right)}{d\eta^2}+\frac{d^2\left(\frac{r\mathrm{V}}{a}\right)}{d\zeta^2}\right\},$$

which, for brevity's sake, may be written

$$\nabla^2\mathrm{V} = \frac{a^5}{r^5}\nabla_1^2\left(\frac{r\mathrm{V}}{a}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

whence, if $\nabla^2\mathrm{V} = 0$, $\qquad \nabla_1^2\left(\frac{r\mathrm{V}}{a}\right) = 0,$

which proves the proposition.

When we have occasion to distinguish the two solutions, we shall use the symbol V′ for the second.

It is to be observed that at the surface of the sphere $r = a$ we have

$$\mathrm{V}' = \mathrm{V}.$$

2. In order to find out the significance of the secondary solution proved in the last Article, let us find out what it means in the case of a single particle of mass m placed at the point (α, β, γ). The potential at (x, y, z) due to m is

$$\frac{m}{\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}}.$$

Therefore, by § 1, La Place's equation is satisfied by

$$V = \frac{a}{r} \frac{m}{\sqrt{\left(\frac{a^2}{r^2}x - \alpha\right)^2 + \left(\frac{a^2}{r^2}y - \beta\right)^2 + \left(\frac{a^2}{r^2}z - \right)^2}}$$

$$= \frac{ma}{\sqrt{a^4 - 2a^2(ax+\beta y+\gamma z) + (\alpha^2+\beta^2+\gamma^2)(x^2+y^2+z^2)}}.$$

If we take a new point $(\alpha', \beta', \gamma')$ such that

$$\alpha' = \frac{a^2\alpha}{\alpha^2+\beta^2+\gamma^2}, \quad \beta' = \frac{a^2\beta}{\alpha^2+\beta^2+\gamma^2}, \quad \gamma = \frac{a^2\gamma}{\alpha^2+\beta^2+\gamma^2},$$

and put

$$\alpha^2+\beta^2+\gamma^2 = f^2,$$
$$\alpha'^2+\beta'^2+\gamma'^2 = f'^2,$$

the above expression for the potential becomes

$$\frac{m\frac{a}{f}}{\sqrt{(x-\alpha')^2+(y-\beta')^2+(z-\gamma')^2}}.$$

The result is therefore the potential at (x, y, z) due to a mass $\frac{m'}{f}$ situated at $(\alpha', \beta', \gamma')$. The point $(\alpha', \beta', \gamma')$ is called the image of the point (α, β, γ) in the sphere whose centre is the origin and whose radius is a.

Since $V' = V$ along the sphere $r = a$, we see that, if that sphere is the surface of a conductor, the electricity induced on it by an electrified point m at (α, β, γ) will be $-\frac{ma}{f}$, and the potential due to the induced electricity at external points will be the same as if it were collected at $(\alpha', \beta', \gamma')$.

3. The result proved in § 2 leads to one or two propositions in the subject of attractions. Thus, let

$$\iiint \frac{\rho\, d\alpha\, d\beta\, d\gamma}{\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}} = \phi(x, y, z)$$

be the known potential at x, y, z due to any gravitating mass; the integrations being performed over the whole volume of the mass. Then, since

$$\frac{d\alpha\, d\beta\, d\gamma}{d\alpha' d\beta' d\gamma'} = \frac{f^3}{f'^3} = \frac{\alpha^6}{f'^6},$$

when we perform the operation of § 1 on the above, recollecting § 2, we get

$$\iiint \frac{\rho \frac{a^5}{f'^5} d\alpha' d\beta' d\gamma'}{\sqrt{(x-\alpha')^2+(y-\beta')^2+(z-\gamma')^2}} = \frac{a}{r}\phi\left(\frac{a^2}{r^2}x, \frac{a^2}{r^2}y, \frac{a^2}{r^2}z\right),$$

the integrations being now performed over the whole volume of the image, a result which might have been anticipated from equation (2), if we express Poisson's as well as La Place's equation in the new co-ordinates.

We can thus, from the known potential at any point due to a given mass, deduce the potential at any point due to its image in the sphere of inversion, the volume density ρ becoming at the corresponding point of the image $\rho \frac{a^5}{f'^5}$.

Similarly, we can deduce from the known potential of a shell of matter that due to its image in the sphere of inversion, the surface density σ becoming at the corresponding point of the image $\sigma \frac{a^3}{f'^3}$.

4. A proposition, corresponding to the above, may be established for the equation in two dimensions, but of a more general character. For if α, β be any two conjugate functions so that

$$\alpha+\beta\sqrt{-1} = f(x+y\sqrt{-1}),$$

then $$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2} = f'(x+y\sqrt{-1})\,f'(x-y\sqrt{-1})\left(\frac{d^2V}{d\alpha^2}+\frac{d^2V}{d\beta^2}\right).$$ *

The interpretation, however, is not so easy except when

$$\alpha-\beta\sqrt{-1} = \frac{a^2}{x+y\sqrt{-1}},$$

i. e., when

$$\alpha = \frac{a^2}{x^2+y^2}x,$$

$$\beta = \frac{a^2}{x^2+y^2}y.$$

In that case, if $F(V, x, y) = 0$ be any solution of La Place's equation, so also is

$$F\left(V, \frac{a^2}{x^2+y^2}x, \frac{a^2}{x^2+y^2}y\right) = 0.$$

To interpret this result, let A be the point where an infinite line perpendicular to the plane of xy cuts that plane. Then the potential at any point P in the plane due to this line may be written

$$g+2E\log\frac{b}{AP} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where g is the potential at distance b from A.

If A′ is the image of A in a circle, centre the origin and radius a,

* Clerk-Maxwell, "Electricity and Magnetism," Vol. i., Theorem iii., on Conjugate Functions.

and if AO $= f$, then the effect of putting $\frac{a^2x}{x^2+y^2}$ for x, and $\frac{a^2y}{x^2+y^2}$ for y, in the Cartesian expression for AP, will be to change it into $\frac{\text{A}'\text{P}}{\text{OP}}f$. The quantity (3) will therefore become

$$\text{g}+2\text{E}\log\left(\frac{b}{\text{A}'\text{P}}\times\frac{\text{OP}}{f}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4),$$

i. e.,

$$\left\{\text{g}+2\text{E}\log\left(\frac{ba}{f.\,\text{A}'\text{P}}\right)\right\}+2\text{E}\log\frac{\text{OP}}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Now, if the circle above referred to be the cross section of an infinite conducting cylinder, the first part { } of (5), with its sign changed, is the potential of the electricity induced in the cylinder by the outside line. In like manner, the same thing would be true for any other electrified line parallel to the conductor. Suppose, therefore, any cylindrical electrification placed in front of the conductor, if we know its potential for outside points, the above operation will produce a result which consists of two parts, viz. :—(1) minus the potential of the induced electricity; (2) minus the potential due to an electrification equal to that outside collected on a line through the origin. The density of the image will vary inversely as the fourth power or as the square of the distance from the origin, according as volume or surface density is under consideration.

5. It may be interesting to notice, in passing, a more general transformation of La Place's equation, of which that in § 1 is a particular case.

Let α, β be any two conjugate functions, so that

$$\alpha+\beta j = f(x+yj),$$

where $j=\sqrt{-1}$. Then, if we put

$$2\xi = 2\,r\alpha_1 = f\left(\frac{x+yj}{r^2}\right)+f\left(\frac{x-yj}{r^2}\right),$$

$$2\eta = 2jr\beta_1 = f\left(\frac{x+yj}{r^2}\right)-f\left(\frac{x-yj}{r^2}\right),$$

$$\zeta = r\frac{z}{r^3} = \frac{z}{r^2},$$

it is obvious that α_1, β_1, and $\frac{z}{r^3}$ satisfy La Place's equation, and that ξ, η are the parameters of two rings cutting one another orthogonally, as well as the sphere whose parameter is ζ. The general transformation for orthogonal surfaces becomes in this case

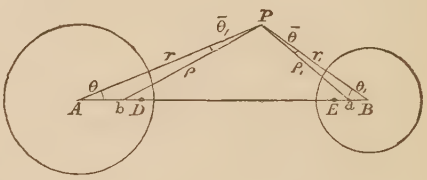
$$\nabla^2\text{V} = \frac{1}{r^5}\left\{f'\left(\frac{x+yj}{r^2}\right)f'\left(\frac{x-yj}{r^2}\right)\left(\frac{d^2(r\text{V})}{d\xi^2}+\frac{d^2(r\text{V})}{d\eta^2}\right)+\frac{d^2(r\text{V})}{d\zeta^2}\right\},$$

the radius of the sphere of inversion being here, for simplicity, equal to unity.

There does not seem, however, to be any result deducible from this which may not equally well be derived from the more simple case in § 1.

The above method will now be applied to one or two cases.

6. Let A and B be the centres of two spherical conductors whose radii are a and b, and whose centres are at a distance c apart. Let a be the geometrical image of the point A in the sphere B so that $Ba = \frac{b^2}{c}$; and, similarly, let b be the image of B in the sphere A so that $Ab = \frac{a^2}{c}$. It will be convenient to specify here some of the symbols to be employed. If P be any point, we shall put for AP, BP, bP, aP, $P\widehat{A}B$, $P\widehat{B}A$, $A\widehat{P}b$, aPB, $\cos\theta$, the symbols



$$r, \quad r_1, \quad \rho, \quad \rho_1, \quad \theta, \quad \theta_1, \quad \bar{\theta}_1, \quad \bar{\theta}, \quad \mu.$$

Let the potential at any point in the interior of A due to its own electrification be given by

$$\phi\left(\mu, \frac{r}{a}\right) = A_0P_0 + A_1P_1\frac{r}{a} + A_2P_2\left(\frac{r}{a}\right)^2 + \ldots\ldots,$$

where P_0, P_1, ... are zonal harmonics whose axis is the line of centres.

The potential due to the electrification of A at any point outside of it will be

$$\frac{a}{r}\,\phi\left(\mu, \frac{a}{r}\right).$$

The charge will be A_0a; and the density, σ, at any point is easily found from the equation

$$\frac{dV}{dr} + \frac{dV'}{dr'} + 4\pi\sigma = 0.$$

In like manner, let the potential at any point inside of B due to B's electrification be

$$\chi\left(\mu_1, \frac{r_1}{b}\right) = B_0Q_0 + B_1Q_1\frac{r_1}{b} + B_2Q_2\left(\frac{r}{b}\right)^2 + \ldots\ldots$$

The potential outside will be

$$\frac{b}{r_1}\,\chi\left(\mu_1, \frac{b}{r_1}\right),$$

and the charge will be B_0b.

If the potentials of the spheres be A and B, we therefore have, for all points inside A,

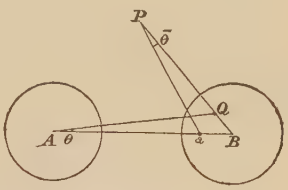
$$\phi\left(\mu, \frac{r}{a}\right) + \frac{b}{r_1}\chi\left(\mu_1, \frac{b}{r_1}\right) = \mathrm{A} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\alpha).$$

Similarly, for all points inside B,

$$\frac{a}{r}\phi\left(\mu, \frac{a}{r}\right) + \chi\left(\mu_1, \frac{r_1}{b}\right) = \mathrm{B} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\beta).$$

7. The potential due to A at a point Q within B is, as we have seen,

$$\frac{a}{r}\phi\left(\mu, \frac{a}{r}\right) = \mathrm{V}_{\mathrm{A,Q}}.$$



Performing on $\mathrm{V}_{\mathrm{A,Q}}$ the operation of § 1 with respect to B as the centre of inversion, we get the potential at a point outside of B, due to the image of A in B with sign reversed. This potential is $\mathrm{V}_{\mathrm{AB,P}} = -\frac{b}{r_1}\mathrm{V}'_{\mathrm{A,Q}}$, where $\mathrm{V}'_{\mathrm{A,Q}}$ is to be derived from $\mathrm{V}_{\mathrm{A,Q}}$ by putting $\bar{\theta}$ for θ, and $\frac{c\rho_1}{r_1}$ for r. We then get

$$\mathrm{V}_{\mathrm{AB,P}} = -\frac{ab}{c\rho_1}\phi\left(\bar{\mu}, \frac{ar_1}{c\rho_1}\right).$$

Similarly, the potential due to the image of B in A is given by

$$\mathrm{V}_{\mathrm{BA,P}} = -\frac{ab}{c\rho}\chi\left(\bar{\mu}_1, \frac{br}{c\rho}\right).$$

Now we may consider the potential at external points due to the electrification of A as made up of two parts, viz., one part due to a uniform distribution and equal to $\frac{\mathrm{A}a}{r}$, and the other due to the electricity induced in A by B, and amounting to the expression just found with its sign reversed. Therefore, outside A, we have

$$\frac{a}{r}\phi\left(\mu, \frac{a}{r}\right) + \frac{ab}{c\rho}\chi\left(\bar{\mu}_1, \frac{br}{c\rho}\right) = \frac{\mathrm{A}a}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\gamma).$$

In like manner, at all points outside of B,

$$\frac{b}{r_1}\chi\left(\mu_1, \frac{b}{r_1}\right) + \frac{ab}{c\rho_1}\phi\left(\bar{\mu}, \frac{ar_1}{c\rho_1}\right) = \frac{\mathrm{B}b}{r_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\delta).$$

8. Subtracting (δ) and (α), we get, for all points inside the sphere A,

$$\phi\left(\mu, \frac{r}{a}\right) - \frac{ab}{c\rho_1}\phi\left(\bar{\mu}, \frac{ar_1}{c\rho_1}\right) = \mathrm{A} - \mathrm{B}\frac{b}{r_1} \quad \ldots\ldots\ldots\ldots\ldots (\epsilon).$$

In like manner, for all points inside of B,

$$\chi\left(\mu_1, \frac{r_1}{b}\right) - \frac{ab}{c\rho}\chi\left(\overline{\mu_1}, \frac{br}{c\rho}\right) = \mathrm{B} - \mathrm{A}\frac{a}{r} \quad \ldots\ldots\ldots\ldots\ldots (\zeta).$$

The quantities $\overline{\mu}$, r_1, ρ_1 in equation (ϵ) are functions of r and μ, so that we may regard that equation as a functional equation in r and μ for the determination of ϕ. From the extremely complicated nature of the function expressing the image, it seems hopeless to attempt any other solution than in some sort of series. Meantime, however, we remark that, if the spheres are at some distance apart, as compared to their radii, a very easy solution is obtained by expanding everything in zonal harmonics and equating coefficients. For the greatest value of $\overline{\theta}$, inside of A, is

$$\tan^{-1}\frac{ab^2}{(c^2-b^2)(c^2-a^2)^{\frac{1}{2}}};$$

and if $\frac{a}{c}$, $\frac{b}{c}$ be not large fractions, the cosine of θ will differ from unity by a quantity depending on the sixth powers of these fractions. Again, for any small angle $\delta\theta$ the zonal harmonic of i^{th} degree becomes

$$1 - \frac{i(i+1)}{2}(\delta\theta)^2 + \frac{3i(i-1)(i^2+3i+4)+4i(i+1)}{8\,\lfloor 4}(\delta\theta)^4 - \&c.,$$

which may be taken, for the angle now thought of, equal to unity. Each term of $\frac{ab}{c\rho_1}\phi\left(1, \frac{ar_1}{c\rho_1}\right)$ may now be expanded as far as the first power of r; and $\frac{1}{r_1}$ being also expanded, we can now equate powers of r and coefficients of the harmonics, and all the unknown coefficients A_0, A_1, ... will be determined. We shall not, however, pursue this approximate solution, but proceed at once to the general case.

9. If we put

$$\phi\left(\mu, \frac{r}{a}\right) = \frac{\mathrm{K}}{\sqrt{(\lambda a)^2 - 2\lambda a r\mu + r^2}},$$

where λ may have any numerical value, then, recollecting that

$$r_1^2 = c^2 - 2cr\mu + r^2,$$

$$\rho_1^2 = \left(c - \frac{b^2}{c}\right)^2 - 2\left(c - \frac{b^2}{c}\right)r\mu + r^2,$$

and

$$\overline{\mu} = \frac{r_1^2 + \rho_1^2 - \frac{b^4}{c^2}}{2r_1\rho_1},$$

we get

$$\frac{ab}{c\rho_1}\phi\left(\bar{\mu}, \frac{ar_1}{c\rho_1}\right)$$

$$= \frac{Kb}{\sqrt{\{\lambda(c^2-b^2)-ac\}^2 - 2(\lambda c-a)\{\lambda(c^2-b^2)-ac\}r\mu + (\lambda c-a)^2 r^2}}$$

$$= \frac{K\dfrac{b}{\lambda c-a}}{\sqrt{(\lambda_1 a)^2 - 2\lambda_1 a r\mu + r^2}},$$

where $$\lambda_1 a = \frac{\lambda(c^2-b^2)-ac}{\lambda c-a}.$$

We may state this result briefly thus:—If we put $\phi\left(\mu, \frac{r}{a}\right) = \frac{K}{\rho}$ where ρ is the distance of any point P from a fixed point on the line of centres at distance λa from the centre of A, then

$$\frac{ab}{c\rho_1}\phi\left(\bar{\mu}, \frac{ar_1}{c\rho_1}\right) = \frac{Kb}{\lambda c-a}\frac{1}{\rho_1},$$

where, in the meantime, ρ_1 denotes the distance of P from a fixed point on the line of centres at distance $\lambda_1 a$ from the centre of A, λ_1 being connected with λ by the above relation.

10. So far as the first term A on the second side of equation (ϵ) is concerned, we therefore have

$$\phi\left(\mu, \frac{r}{a}\right) = Aa\left(\frac{1}{a} + \frac{b}{c}\frac{1}{\rho_1} + \frac{b^2}{c(\lambda_1 c-a)}\frac{1}{\rho_2} + \ldots\ldots\right.$$
$$\left. \ldots\ldots + \frac{b^n}{c(\lambda_1 c-a)\ldots(\lambda_{n-1}c-a)}\frac{1}{\rho_n} + \ldots\ldots\right),$$

in which it is seen that the $\phi\left(\mu, \frac{r}{a}\right)$ of any term strikes out with the $\frac{ab}{c\rho_1}\phi\left(\bar{\mu}, \frac{ar_1}{c\rho_1}\right)$ of its predecessor, and the question now comes to be how to determine from the law of formation of the quantities λ the general expression for any one of them. The quantity ρ_n is then found, being $\sqrt{(a\lambda_n)^2 - 2a\lambda_n r\mu + r^2}$.

We have $$\lambda_{n+1} = \frac{c}{a} - \frac{\lambda_n b^2}{a(\lambda_n c-a)}.$$

Let $$\lambda_n c - a = \xi_n.$$

Then $$a\xi_n\xi_{n+1} - (c^2-a^2-b^2)\xi_n + ab^2 = 0.$$

Put $$\xi_n = \eta_n + v,$$

and let $$av^2 - (c^2-a^2-b^2)v + ab^2 = 0;$$

we get, after some simplifications,

$$\frac{1}{\eta_{n+1}} - \frac{v^2}{b^2}\frac{1}{\eta_n} = \frac{v}{b^2}.$$

The solution of this is

$$\frac{1}{\eta_n} = \mathrm{C}\left(\frac{v}{b}\right)^{2n} + \frac{v}{b^2-v^2}.$$

But when $n=1$, $\eta_1 = \xi_1 - v = \lambda_1 c - a - v$,

in this case $= \dfrac{c^2-b^2}{a} - a - v$

$$= \frac{b^2}{v}.$$

Hence, determining C and transferring to ξ_n, we have

$$\frac{b}{\xi_n} = \frac{v}{b}\,\frac{1-\left(\frac{v}{b}\right)^{2n}}{1-\left(\frac{v}{b}\right)^{2n+2}},$$

and $$\lambda_n a = (\lambda_n c - a)\frac{a}{c} + \frac{a^2}{c} = \frac{a}{c}(\xi_n + a).$$

11. As regards the second term on the right-hand side of (ϵ), a similar investigation applies. The determination of the constant C will be different, the value of λ_1 corresponding to r_1 being $\frac{c}{a}$. Using the same notation as in § 10, we have, in this case,

$$\frac{b}{\xi_n} = \frac{v}{b}\,\frac{1-\left(\frac{c}{a+v}\right)^2\left(\frac{v}{b}\right)^{2n}}{1-\left(\frac{c}{a+v}\right)^2\left(\frac{v}{b}\right)^{2n+2}},$$

and, as before, $$\lambda_n a = \frac{a}{c}(\xi_n + a).$$

If we denote the successive values of $\sqrt{(\lambda_n a)^2 - 2\lambda_n a r\mu + r^2}$ in this case by r_n, then the complete solution of $\phi\left(\mu, \frac{r}{a}\right)$ is

$$\mathrm{A}a\left\{\frac{1}{a} + \frac{b}{c}\frac{1}{\rho_1} + \ldots\ldots + \frac{b}{c}\left(\frac{v}{b}\right)^{n-1}\frac{1-\left(\frac{v}{b}\right)^2}{1-\left(\frac{v}{b}\right)^{2n}}\frac{1}{\rho_n} + \ldots\ldots\right\}$$

$$-\mathrm{B}b\left\{\frac{1}{r_1} + \ldots\ldots + \left(\frac{v}{b}\right)^{n-1}\frac{1-\left(\frac{c}{a+v}\right)^2\left(\frac{v}{b}\right)^2}{1-\left(\frac{c}{a+v}\right)^2\left(\frac{v}{b}\right)^{2n}}\frac{1}{r_n} + \ldots\ldots\right\}.$$

12. Referring to § 10, we had a quadratic equation for the determination of v, the product of the roots being b^2. In the immediately preceding articles, we must therefore adopt the smaller root, in order that the successive powers of $\frac{v}{b}$ may diminish.

If we consider again the law of formation of λ, we see from § 9 that λ will return to itself provided

$$\lambda a = \frac{\lambda(c^2-b^2)-ac}{\lambda c-a},$$

i. e., provided $$\lambda^2 ac - \lambda(c^2-b^2+a^2)+ac = 0.$$

There are therefore two points on the line of centres for which this is true. They are marked D and E in the figure of § 6, and they are such that each is the inverse point of the other in either sphere, as can be easily proved.

Solving the above equation,

$$\lambda a = \frac{c^2-b^2+a^2}{2c} \pm \frac{\sqrt{a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2}}{2c},$$

and putting the quantity after the double sign equal to k, we have

$$\mathrm{DE} = 2k,$$

$$\mathrm{AD} = \frac{c^2-b^2+a^2}{2c} - k,$$

$$\mathrm{AE} = \frac{c^2-b^2+a^2}{2c} + k.$$

The two roots of the quadratic in v are given by

$$v_1 = \frac{c^2-a^2-b^2}{2a} - \frac{c}{a}k,$$

$$v_2 = \frac{c^2-a^2-b^2}{2a} + \frac{c}{a}k;$$

and the values of v are connected with those of λ by the equation

$$v+a = \lambda c.$$

If now R, R_1 denote the distances of any point from D and E, and we put

$$\phi\left(\mu, \frac{r}{a}\right) = \frac{\mathrm{K}}{\sqrt{\mathrm{R}\mathrm{R}_1}},$$

then $\frac{ab}{c\rho_1}\phi\left(\bar{\mu}, \frac{ar_1}{c\rho_1}\right)$ will be

$$\sqrt{\frac{b}{v_1}} \cdot \sqrt{\frac{b}{v_2}} \frac{\mathrm{K}}{\sqrt{\mathrm{R}\mathrm{R}_1}} = \frac{\mathrm{K}}{\sqrt{\mathrm{R}\mathrm{R}_1}};$$

so that this value of $\phi\left(\mu, \frac{r}{a}\right)$ satisfies the left-hand side of the equation. Moreover, K may be any function which does not change its

value when μ becomes $\bar{\mu}$, and $\frac{r}{a}$ becomes $\frac{ar_1}{c\rho_1}$. Inasmuch, however, as $\frac{K}{\sqrt{RR'}}$ becomes infinite inside of the sphere A, it must obviously be discarded as part of the expression for ϕ.

13. The quantity of electricity E_a on the sphere A is to be found by putting $r=0$ in the expression for the potential and multiplying by a. If, following Prof. Maxwell, we put $E_a = q_{aa}A + q_{ab}B$, where q_{aa} is the coefficient of capacity of A, and q_{ab} the coefficient of induction of B on A, then

$$q_{aa} = a + \frac{a^2 b}{c^2 - b^2} + \ldots\ldots,$$

the $(n+1)^{\text{th}}$ term being

$$\frac{abv}{b^2+av}\left(\frac{v}{b}\right)^{n-1}\frac{1-\left(\frac{v}{b}\right)^2}{1-\left(\frac{v+a}{c}\right)^2\left(\frac{v}{b}\right)^{2n}}.$$

In like manner,

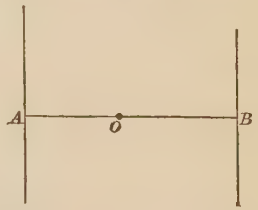
$$-q_{ab} = \frac{ab}{c} + \ldots\ldots,$$

the n^{th} term being

$$\frac{bcv}{b^2+av}\left(\frac{v}{b}\right)^{n-1}\frac{1-\left(\frac{c}{a+v}\right)^2\left(\frac{v}{b}\right)^2}{1-\left(\frac{v}{b}\right)^{2n}}.$$

In like manner, the coefficient of capacity q_{bb} of the sphere B may be found; the coefficient $q_{ba} = q_{ab}$.

14. All these expressions are greatly simplified if the two spheres be in contact. In that case we have $v=b$ and $c=a+b$, and we have to find the values of the vanishing fractions in the expressions just given. It will be more interesting, however, and easier to apply the general method of § 1 coupled with the theory of inversion. Let A and B be two parallel infinite plane conductors, and let an electrified point c be placed between them, at distances $\frac{1}{2a}$ and $\frac{1}{2b}$; and for brevity's sake, let $\frac{1}{2a}+\frac{1}{2b}=c$. Let $\phi(x,y)$ denote the potential due to the electrification on A at any point whose distance from A is x and distance from the axis AB is y. Similarly, let $\chi(x,y)$ denote a similar quantity due to B's electrification. The image of A in B will be at a distance c to the right of B, and the potential due to it will be



of the same form as for A itself, x then signifying distance from the image. In like manner, we can find the potential due to the image of B in A. Let us consider only points on AB, or AB produced either way, and let $\phi(x, 0)$ and $\chi(x, 0)$ be written $\phi(x)$ and $\chi(x)$. If we now take A for origin and measure all the distances from A, we shall have for any point—

1°. To the left of A: $\phi(x)+\chi(x+c)+\dfrac{e}{\dfrac{1}{2a}+x}=0$ ..... (α).

2°. To the right of B: $\phi(x)+\chi(x-c)+\dfrac{e}{x-\dfrac{1}{2a}}=0$ ..... (β).

3°. To the right of A: $\phi(x)+\chi(x+c)+\dfrac{e}{x+c}=0$ ..... (γ).

4°. To the left of B: $\phi(x+2c)+\chi(x+c)+\dfrac{e}{x+c+\dfrac{1}{2b}}=0$.....$(\delta)$.

The last equation is written in a form suitable for points to the left of A. For points between A and B the equation would be

$$\phi(-x+2c)+\chi(-x+c)+\frac{e}{-x+c+\dfrac{1}{2b}}=0.$$

Subtracting (δ) from (α), we have, for points to the left of A,

$$\phi(x)-\phi(x+2c)+\frac{e}{\dfrac{1}{2a}+x}-\frac{e}{c+\dfrac{1}{2b}+x}=0.$$

The solution of this equation is

$$-\frac{e}{2c}\int_0^1\frac{\theta^{\frac{1}{2c}\left(\frac{1}{a}+x\right)-1}-1}{1-\theta}d\theta+\frac{e}{2c}\int_0^1\frac{\theta^{\frac{1}{2c}\left(c+\frac{1}{2b}+x\right)-1}-1}{1-\theta}d\theta$$

$$=-\frac{e}{2c}\int_0^1\frac{\theta^{\frac{1}{2c}\left(\frac{1}{2a}+x\right)-1}\left(1-\theta^{\frac{1}{2bc}}\right)}{1-\theta}d\theta.$$

The complementary function, which is a constant, is in this case zero, because, if e were zero, ϕ would be also zero.

If we now transfer the origin to the point O, the expression just found becomes

$$-\frac{e}{2c}\int_0^1\frac{\theta^{\frac{x}{2c}-1}\left(1-\theta^{\frac{1}{2bc}}\right)}{1-\theta}d\theta.$$

Now invert with respect to a sphere whose centre is the origin and radius unity. If the potential of the sphere into which the plane A

inverts be A, then $e = A$, and we have for all points on the axis inside that sphere the potential due to its electrification equal to

$$\frac{A}{2cx}\int_0^1 \frac{\theta^{\frac{1}{2cx}-1}(1-\theta^{\frac{1}{2bc}})}{1-\theta}\,d\theta.$$

If we transfer the origin again, this time to the centre of the sphere, we shall have to change x into $a+x$. Doing so, and then putting $x=0$, we find for the charge of the sphere, $2c$ being $\frac{1}{a}+\frac{1}{b}$,

$$A\,\frac{ab}{a+b}\int_0^1 \frac{\theta^{\frac{b}{a+b}-1}-1}{1-\theta}\,d\theta,$$

which is the form found by Poisson.

Mr. J. W. L. Glaisher has pointed out that this integral can be expressed at once in terms of a function which has been tabulated by Gauss ("Werke," Band iii., pp. 161, 162).

Putting $$\int_0^1 \frac{\theta^z-1}{1-\theta}\,d\theta = -\gamma-\psi(z),$$

where γ is Euler's constant ·5771215, we find $\psi(z)$ tabulated for a series of values of z between 0 and 1.

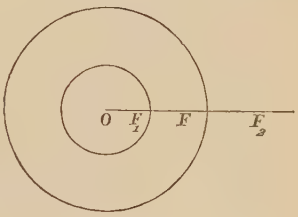
Now $$\int_0^1 \frac{\theta^{z-1}-\theta^z}{1-\theta}\,d\theta = \frac{1}{z}.$$

Hence $$E_a = Aa\left\{1-\frac{b}{a+b}\left[\gamma+\psi\left(\frac{b}{a+b}\right)\right]\right\}.$$

It may be interesting to tabulate, for different values of $\frac{b}{a}$, a few of the values of the ratio $\frac{E_a}{Aa}$, that is, the ratio of the charge of the sphere A at the potential A, when in contact with B, to its charge when it is by itself at the same potential.

$\frac{b}{a}$	$\frac{E_a}{Aa}$	$\frac{b}{a}$	$\frac{E_a}{Aa}$
99	·0164	$\frac{2}{3}$	·7937
9	·1698	$\frac{3}{7}$	·8776
4	·3102	$\frac{1}{4}$	·9424
$\frac{7}{3}$	·4500	$\frac{1}{9}$	·9847
$\frac{3}{2}$	·5780	$\frac{1}{99}$	·9984
1	·6931		

15. There are but few cases which will yield to this method, and the next we shall apply it to has been already worked out, though in an entirely different way, in Clerk-Maxwell's "Electricity and Magnetism." It is obvious, however, from the work of the last Article, that we could solve in a simple form the case of two spheres in contact, under the influence of any number of outside electrified points in the line of centres. Without, however, going into that, let us now take the case of two uninsulated concentric spheres with an electrified point e placed at a point F between them at a distance f from the common centre O. Let a be the radius of the smaller, and β of the larger; also let F_1 and F_2 be the images of F in these two respectively. Then, if r be any radius vector from O, and μ the cosine of its inclination to OF, and if $\phi\left(\mu, \frac{r}{a}\right)$, $\chi\left(\mu, \frac{r}{\beta}\right)$ represent the potentials due to the electrifications of the two spheres for points inside each respectively, then for any point P we have



(1) inside A: $$\phi\left(\mu, \frac{}{a}\right) + \chi\left(\mu, \frac{r}{\beta}\right) + \frac{e}{\mathrm{FP}} = 0 \quad \ldots\ldots(\alpha).$$

(2) outside B: $$\frac{a}{r}\phi\left(\mu, \frac{a}{r}\right) + \frac{\beta}{r}\chi\left(\mu, \frac{\beta}{r}\right) + \frac{e}{\mathrm{FP}} = 0 \quad \ldots\ldots(\beta).$$

(3) outside A: $$\frac{a}{r}\phi\left(\mu, \frac{a}{r}\right) + \frac{a}{r}\chi\left(\mu, \frac{a^2}{\beta r}\right) + \frac{ea}{f \cdot \mathrm{F_1P}} = 0 \quad \ldots\ldots(\gamma).$$

(4) inside B: $$\frac{a}{\beta}\phi\left(\mu, \frac{ar}{\beta^2}\right) + \chi\left(\mu, \frac{r}{\beta}\right) + \frac{e\beta}{f \cdot \mathrm{F_2P}} = 0 \quad \ldots\ldots(\delta).$$

If now we subtract (δ) from (α), we get

$$\phi\left(\mu, \frac{r}{a}\right) - \frac{a}{\beta}\phi\left(\mu, \frac{ar}{\beta^2}\right) + \frac{e}{\mathrm{FP}} - \frac{e\beta}{f \cdot \mathrm{F_2P}} = 0, \quad \ldots\ldots$$

for all points inside of A.

In like manner, for all points outside of B,

$$\frac{\beta}{r}\chi\left(\mu, \frac{\beta}{r}\right) - \frac{a}{r}\chi\left(\mu, \frac{a^2}{\beta r}\right) + \frac{e}{\mathrm{FP}} - \frac{ea}{f \cdot \mathrm{F_1P}} = 0.$$

If we expand ϕ and χ in La Place's functions, and equate coefficients, we find

$$\phi\left(\mu, \frac{r}{a}\right) = -\frac{e}{f}\left[\frac{\beta - f}{\beta - a} + \frac{\beta^3 - f^3}{\beta^3 - a^3}\mathrm{P}_1\frac{r}{f} + \ldots + \frac{\beta^{2n+1} - f^{2n+1}}{\beta^{2n+1} - a^{2n+1}}\mathrm{P}_n\left(\frac{r}{f}\right)^n + \ldots\right],$$

$$\chi\left(\mu, \frac{r}{\beta}\right) = -\frac{e}{f}\left[\frac{f - a}{\beta - a} + \frac{f^3 - a^3}{\beta^3 - a^3}\mathrm{P}_1\frac{r}{f} + \ldots + \frac{f^{2n+1} - a^{2n+1}}{\beta^{2n+1} - a^{2n+1}}\mathrm{P}_n\left(\frac{r}{f}\right)^n + \ldots\right].$$

The charges induced in the two spheres are therefore

$$-\frac{a}{f}\frac{\beta-f}{\beta-a}e, \quad -\frac{\beta}{f}\frac{f-a}{\beta-a}e.$$

16. If we now invert with regard to the point F, making the radius of inversion f, the figure of § 15 will become that of § 6, F and O becoming the two points E and D of that figure. We have the following relations for the radii of the two concentric spheres, if we recollect that f and $2k$ are the same thing:

$$a = 2k \times \frac{\mathrm{AD}}{a},$$

$$\beta = 2k \times \frac{\mathrm{BD}}{b}.$$

Now, according to the operation of § 1, $\frac{a}{r}\phi\left(\mu, \frac{a}{r}\right)$ will become

$$\frac{2k}{\mathrm{R}_1}\frac{a}{2k}\frac{\mathrm{R}_1}{\mathrm{R}}\phi\left(\bar{\mu}, \frac{a}{2k}\frac{\mathrm{R}_1}{\mathrm{R}}\right)$$

$$= \frac{a}{\mathrm{R}}\phi\left(\bar{\mu}, \frac{a}{2k}\frac{\mathrm{R}_1}{\mathrm{R}}\right)$$

$$= \frac{2k\lambda_1}{\mathrm{R}}\phi\left(\bar{\mu}, \lambda_1\frac{\mathrm{R}_1}{\mathrm{R}}\right),$$

R, R_1 being, as before, the distances of any point from D and E. The quantity $\bar{\mu}$ is the cosine of the angle subtended by DE at the point whose distances are R, R_1. In fact, $\bar{\mu}$ and $\frac{\mathrm{R}_1}{\mathrm{R}}$ may be taken as the dipolar coordinates of the point; and the solution which we have just found, *i.e.*, when we have made the changes described on the two sides of the equations in the last article, is really what we should obtain by the solution of the dipolar coordinate form of La Place's equation, given in Sir W. Thomson's Reprint of Papers, pp. 147, 148. It is to be remarked, however, in this case that the inverted spheres are at the same potentials. It is clear, however, from our previous solution, §§ 6—13, that, of the two series into which ϕ may be split, one positive and the other negative, when the operation of § 1 is performed on it, the second part, which is the part depending on the sphere B, will only have to be altered by a multiplier depending on the potential of that sphere.

17. If, in the functional equation for ϕ in § 15, we consider only points on OF, and put $\phi\left(1, \frac{x}{a}\right) = \mathrm{F}\frac{x}{a}$, we come to

$$\mathrm{F}\left(\frac{x}{a}\right) - \frac{a}{\beta}\mathrm{F}\left(\frac{ax}{\beta^2}\right) + \frac{e}{f-x} - \frac{e\beta}{\beta^2-fx} = 0.$$

If there were an easy way of solving this equation, we should thereby obtain the best solution of our problem in the same manner as in the case of the two spheres in contact. There seems no way, however, of solving it except by the summation of the series for $\phi\left(1, \frac{x}{a}\right)$ according to the method given by Poisson, a brief account of which is contained in the next article.

18. Let it be required to find the sum of

$$\Sigma_{n=0}^{n=\infty} \frac{\gamma^n}{1+k\phi^{2n}},$$

where γ and ϕ are any quantities whatever, and k may be positive or negative.

We begin with

$$4\int_0^\infty \frac{\sin p\theta}{e^{\pi\theta}-e^{-\pi\theta}}\,d\theta = \frac{e^p-1}{e^p+1} = 1-\frac{2}{e^p+1},$$

a result which is given by Bertrand, "Calcul Intégral," p. 158.

Now let $\phi = e^\xi$, so that $\log\phi = \xi$.

Also let $$p = 2n\xi + \log k.$$

Then $$\frac{1}{1+k\phi^{2n}} = \frac{1}{1+e^p} = \frac{1}{2} - 2\int_0^\infty \frac{\sin p\theta}{e^{\pi\theta}-e^{-\pi\theta}}\,d\theta;$$

therefore $$\frac{\gamma^n}{1+k\phi^{2n}} = \frac{\gamma^n}{2} - 2\int_0^\infty \frac{\gamma^n \sin(2n\xi+\log k)\,\theta}{e^{\pi\theta}-e^{-\pi\theta}}\,d\theta.$$

But $\Sigma_0^\infty \gamma^n \sin(2n\xi+\log k)\,\theta$

$$= \frac{e^{\theta\log k\sqrt{-1}}}{2\sqrt{-1}\,(1-\gamma e^{2\xi\theta\sqrt{-1}})} - \frac{e^{-\theta\log k\sqrt{-1}}}{2\sqrt{-1}\,(1-\gamma e^{-2\xi\theta\sqrt{-1}})}$$

$$= \frac{\sin(\log k)\,\theta - \gamma\sin(\log k - 2\xi)\,\theta}{1-2\gamma\cos 2\xi\theta+\gamma^2}.$$

Hence

$$\Sigma_0^\infty \frac{\gamma^n}{1+k\phi^{2n}} = \frac{1}{2(1-\gamma)} - 2\int_0^\infty \frac{\sin(\log k)\,\theta - \gamma\sin(\log k-2\xi)\,\theta}{(e^{\pi\theta}-e^{-\pi\theta})(1-2\gamma\cos 2\xi\theta+\gamma^2)}\,d\theta.$$

Now, instead of k, put $k(\cos\psi+\sqrt{-1}\sin\psi)$, and equate rational and irrational parts. We shall find

$$\int_0^\infty \frac{\cos(\log k)\,\theta - \gamma\cos(\log k-2\xi)\,\theta}{1-2\gamma\cos 2\xi\theta+\gamma^2}\,d\theta = 0.$$

And, putting $\psi = \pi$,

$$\Sigma_0^\infty \frac{\gamma^n}{1-k\phi^{2n}}$$

$$= \frac{1}{2(1-\gamma)} - \int_0^\infty \frac{e^{\pi\theta}+e^{-\pi\theta}}{e^{\pi\theta}-e^{-\pi\theta}} \cdot \frac{\sin(\log k)\,\theta - \gamma\sin(\log k-2\xi)\,\theta}{1-2\gamma\cos 2\xi\theta+\gamma^2}\,d\theta.$$

Two other expressions for the same thing are proved by Poisson, their application depending on the convergency of the series; these we shall merely state. They are

$$\frac{1}{2(1-\gamma)}+\int_0^\infty \frac{e^{(\log k)\theta}\,d\theta}{1-\gamma e^{2\xi\theta}}-2\int_0^\infty \frac{\sin(\log k)\theta-\gamma\sin(\log k-2\xi)\theta}{(e^{2\pi\theta}-1)(1-2\gamma\cos 2\xi\theta+\gamma^2)}\,d\theta,$$

and

$$\frac{1}{2(1-k)}+\int_0^\infty \frac{e^{(\log\gamma)\theta}\,d\theta}{1-k e^{2\xi\theta}}-2\int_0^\infty \frac{\sin(\log\gamma)\theta-\gamma\sin(\log\gamma-2\xi)\theta}{(e^{2\pi\theta}-1)(1-2k\cos 2\xi\theta+k^2)}\,d\theta.$$

19. Let us now use the first of these three results in finding $\phi\left(1,\frac{x}{a}\right)$ in the form of a definite integral. We have

$$-\phi\left(1,\frac{x}{a}\right)=\frac{e}{f}\Sigma_0^\infty\frac{\left(\frac{x}{f}\right)^n}{1-\frac{a}{\beta}\left(\frac{a}{\beta}\right)^{2n}}-\frac{e}{\beta}\Sigma_0^\infty\frac{\left(\frac{xf}{\beta^2}\right)^n}{1-\frac{a}{\beta}\left(\frac{a}{\beta}\right)^{2n}}$$

$$=\frac{e}{2(f-x)}-e\int_0^\infty\frac{e^{\pi\theta}+e^{-\pi\theta}}{e^{\pi\theta}-e^{-\pi\theta}}\cdot\frac{(f+x)\sin\left(\log\frac{a}{\beta}\right)\theta}{f^2-2fx\cos 2\left(\log\frac{a}{\beta}\right)\theta+x^2}\,d\theta$$

$$-\frac{e\beta}{2(\beta^2-fx)}+e\beta\int_0^\infty\frac{e^{\pi\theta}+e^{-\pi\theta}}{e^{\pi\theta}-e^{-\pi\theta}}\cdot\frac{(\beta^2+fx)\sin\left(\log\frac{a}{\beta}\right)\theta}{\beta^4-2\beta^2fx\cos 2\left(\log\frac{a}{\beta}\right)\theta+f^2x^2}\,d\theta.$$

To pass from this case to the general case of two spheres in presence of one another, we must go through the following steps:—1°. Change the origin to F; this converts $f-x$ into x. 2°. Invert at the new origin with regard to a sphere of radius f, or, which is the same thing, $2k$. This converts x into $\frac{4k^2}{x}$. We must also multiply the whole result by $\frac{2k}{x}$. 3°. We must now transfer the origin of coordinates to the centre of the sphere A in § 6. This makes x become $AE-x$. 4°. To get the charge on A, we put $x=0$ and multiply by a. This system of changes amounts to changing the x in the above expression into

$$2k-\frac{4k^2}{AE}=2k\frac{AD}{AE}=2k\frac{\lambda_1}{\lambda_2}=2k\lambda_1^2,$$

where λ_1, λ_2 are the roots of the quadratic in § 12.

Before the new expressions for q_{aa}, q_{ab} are written down, it will be well to recall the relations between the radii of the concentric spheres and those into which they have been inverted.

We have $\qquad a=2k\lambda_1;$

and if we denote by $\mu_1 b$ and $\mu_2 b$ the distances of E and D from the centre of B, then $\beta = 2k\mu_2$.

Hence $$\frac{\alpha}{\beta} = \frac{\lambda_1}{\mu_2} = \lambda_1 \mu_1,$$

which can be proved $= \frac{v_1}{b}$, the smaller root of the quadratic in $\frac{v}{b}$. As we have used it hitherto without the suffix, we may still do so. We have then $$\frac{\alpha}{\beta} = \frac{v}{b}.$$

We have therefore

$$q_{aa} = \frac{\alpha}{2} - \alpha \int_0^\infty \frac{e^{\pi\theta} + e^{-\pi\theta}}{e^{\pi\theta} - e^{-\pi\theta}} \frac{(1+\lambda_1^2) \sin\left(\log \frac{v}{b}\right)\theta}{1 - 2\lambda_1^2 \cos 2\left(\log \frac{v}{b}\right)\theta + \lambda_1^4} d\theta,$$

$$-q_{ab} = \frac{k\lambda_1\mu_2}{\mu_2^2 - \lambda_1^2} - \alpha\mu_2 \int_0^\infty \frac{e^{\pi\theta} + e^{-\pi\theta}}{e^{\pi\theta} - e^{-\pi\theta}} \frac{(\mu_2^2 + \lambda_1^2) \sin\left(\log \frac{v}{b}\right)\theta}{\mu_2^4 - 2\lambda_1^2\mu_2^2 \cos 2\left(\log \frac{v}{b}\right)\theta + \lambda_1^4} d\theta.$$

On the Conditions of Perpendicularity in a Parallelepipedal System.

By H. J. Stephen Smith, Savilian Professor of Geometry in the University of Oxford.

[*Read December 14th*, 1876.]

1. The conception of a parallelepipedal system (*i.e.*, of a space divided by three systems of equidistant parallel planes into similar and equal parallelepipeds) which forms the basis of the usually received theory of crystallography,* has also led to important researches in a domain belonging partly to arithmetic and partly to geometry.†

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* See Dr. Leonhard Sohncke, "Die unbegrenzten regelmässigen Punktsysteme als Grundlage einer Theorie der Krystallstruktur," Karlsruhe, 1876, for a different and more general hypothesis as to the ultimate structure of a crystal. See also a memoir by Dr. Sohncke, "Die regelmässigen ebenen Punkt-systeme von unbegrenzter Ausdehnung" in "Borchardt's Journal," Vol. lxxvii. p. 47, and a memoir by M. Camille Jordan "Sur les groupes de mouvements" in the "Annali di Matematica" (Brioschi e Cremona), Vol. ii. p. 167.

† See (1) Gauss' Review of Seeber's "Untersuchungen ueber die Eigenschaften der positiven ternaren quadratischen Formen" in the Göttingen "Gelehrte Anzeige" for 1831, or in "Crelle's Journal," Vol. xx. p. 312; (2) Lejeune Dirichlet, "Ueber die Reduction der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen," "Crelle's Journal," Vol. xl. p. 209; (3) the "Études Crystallographiques" of Auguste Bravais, Paris 1866; (4) A Letter of Eisenstein to M. Charles Hermite

It is the object of the present paper to discuss, more completely than perhaps has been done as yet, the conditions for the perpendicularity of lines and planes in a parallelepipedal system. The arithmetical principles required in the discussion are very simple, and are to be found, for the most part, in articles 1—7 of a memoir on "Linear Systems of Indeterminate Equations and Congruences" in the "Philosophical Transactions" for 1861. The geometrical considerations involved, although well known, are less elementary, and relate to the theory of contravariant systems of cones. For the definition and some of the principal properties of such systems, we may refer to a paper "On some Geometrical Constructions" in the "Proceedings of the London Mathematical Society (Vol. ii., p. 85); the most important case, that of two-fold systems, forms an essential part of the theory of cubic cones or curves; the considerations relating to that theory which we shall have occasion to employ, will be found in the "Introduzione alla Teorica delle Curve Piane" of Professor Cremona, or in the "Treatise on the Higher Plane Curves" of Dr. Salmon.

The results of the present enquiry (which has been undertaken at the request of Professor N. S. Maskelyne, and owes much to his suggestions) are submitted to the Mathematical Society with great diffidence; because, while they do not seem likely to admit of any direct application to the measurement of crystals, there is also some uncertainty as to their exact relation to other parts of crystallographic theory.

2. It is, perhaps, hardly necessary to explain that by a "line of the system" we understand a line joining any two points of a given parallelepipedal system; by a "plane of the system," a plane containing any three points of the system; the "points of the system" being the points of intersection of the three sets of equidistant parallel planes by which the system is defined. For our present purpose it will be sufficient to consider exclusively *origin* lines and planes; *i.e.*, lines and planes passing through a fixed point of the system taken as origin.

Whenever a line of the system is perpendicular to a plane of the system, the system has a certain "symmetry of aspect" with regard to that plane. Let Ω be the plane, and let O be any point of the system lying in it. The planes and lines of the system which pass through O are symmetrically distributed with regard to Ω; but the points of the system are not (in general) symmetrically distributed with regard to

("Journal de Mathématiques," Vol. xvii. p. 473). The researches to which this letter refers are purely arithmetical, but they have an important bearing on a class of questions which are naturally suggested by the present enquiry, and of which the following may be taken as an example, "In what cases can a parallelepipedal system, possessing a spherical symmetry, be regarded as containing a system in which the parallelepipeds are cubes?"

Ω: thus, if OP is any line of the equation, and OQ is the *reflexion* of OP in the plane Ω, OQ is a line of the system, but the points of the system which lie on OQ are not, in general, the reflexions of the points which lie on OP. Hence, while the points of the system are not themselves symmetrically distributed with regard to Ω, the directions in which they would be viewed by an eye situated at O, are symmetrically distributed; and this we may express by saying that the system has a "symmetry of aspect" with regard to Ω. It will be seen—(1) that Ω is, or is not, a plane of absolute symmetry (*i.e.*, a plane of symmetry with regard to the points of the system), according as the point of the system, that lies the nearest to O upon the normal at O, does, or does not, lie upon the nearest plane of the system parallel to Ω; (2) that two parallelepipedal systems may have the same aspect without coinciding. Suppose, for example, that the given parallelepipedal system has a symmetry of aspect with regard to three rectangular planes intersecting in O (this is the case of an "ellipsoidal symmetry" to which we shall presently refer). Let OA, OB, OC be the three lines of intersection of these planes, A, B, C being the points of the system nearest to O on these lines respectively. The four points O, A, B, C determine a new parallelepipedal system (OABC) composed of rectangular parallelepipeds. This system is *contained* in the given system, because every point of (OABC) is a point of the given system; but the two systems are not *equivalent*, because every point of the given system is not a point of (OABC). In general, a certain number of points of the given system lie in each rectangular parallelepiped of (OABC), and are similarly distributed in each of them, though not symmetrically in any one of them. But the "aspect" of the two systems from the point O (or from any point common to both of them) is the same; and the question whether the given system has an "ellipsoidal symmetry of aspect" is the same as the question whether it contains a rectangular parallelepipedal system. Similar considerations apply to every case of symmetry of aspect; and indeed, whenever one of two parallelepipedal systems contains the other, the two systems have the same aspect at any one of their common points.*

As we shall have no occasion to consider planes of absolute symmetry, we shall, henceforward, for the sake of brevity, use the word symmetry in the sense of "symmetry of aspect." Thus any line, and any plane of the system, which are at right angles to one another, are an axis and a plane of symmetry.

* The relation between a contained and a containing parallelepipedal system is the same as the relation between a contained and a containing ternary quadratic form (see Gauss, *loc. cit.*) The problem, "To find the conditions that a given parallelepipedal system should have an ellipsoidal symmetry of aspect," may be stated in a purely arithmetical form as follows:—"To find the conditions that a definite ternary quadratic form (of which the coefficients may be rational or irrational) should contain a form of the type $Ax^2 + By^2 + Cz^2$."

The cases of symmetry, as thus defined, which can present themselves in a parallelepipedal system, are four in number. There is (1) the case of *simple* symmetry, where there is only one axis and one plane of symmetry; and there are three cases of triple symmetry, which may be characterized as (2) the ellipsoidal, (3) the spheroidal, and (4) the spherical. In an ellipsoidal system there are three rectangular planes of the system which are planes of symmetry; in a spheroidal system there is one equatorial plane of symmetry, but every plane of the system at right angles to this plane is also a plane of symmetry; in a system having spherical symmetry, every plane of the system is a plane of symmetry and every line of the system an axis of symmetry. Two simple symmetries cannot coexist without forming a triple symmetry, which is ellipsoidal, if the axis of one of the symmetries lies in the plane of the other, but is spheroidal in every other case; three simple symmetries form an ellipsoidal symmetry, if the three axes are at right angles to one another; a spheroidal symmetry, if one of the axes is at right angles to the plane of the other two, which are not at right angles to one another; a spherical symmetry in every other case. These assertions are, in part, simple consequences of the well-known theorem that if a plane contains two pair of lines of the system at right angles to another, every line of the system in that plane is at right angles to a line of the system; in part, however, their demonstration requires a fuller discussion of the conditions of symmetry, to which we now proceed.

3. *General Conditions of Perpendicularity.*

We begin by recording, in the form which is most suitable for our present purpose, the conditions for the perpendicularity of lines and planes in space referred to oblique axes.

The coordinates of any point being xyz, and XYZ being the angles between the positive directions of the axes, the formula

$$\mathrm{F}(x, y, z) = x^2+y^2+z^2+2yz\cos\mathrm{X}+2xz\cos\mathrm{Y}+2xy\cos\mathrm{Z},$$

which denotes the square of the distance of the point xyz from the origin, supplies a convenient mode of expressing the condition for the perpendicularity of any two lines

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n},$$

$$\frac{x}{l'} = \frac{y}{m'} = \frac{z}{n'};$$

viz., this condition is

$$(1) \ldots\ldots l'\frac{d\mathrm{F}}{dl}+m'\frac{d\mathrm{F}}{dm}+n'\frac{d\mathrm{F}}{dn} = 0,$$

or $$l\frac{d\mathrm{F}'}{dl'} + m\frac{d\mathrm{F}'}{dm'} + n\frac{d\mathrm{F}'}{dn'} = 0,$$

where $$\mathrm{F} = \mathrm{F}(l, m, n), \qquad \mathrm{F}' = \mathrm{F}(l', m', n').$$

Similarly the condition for the perpendicularity of the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n},$$

and the plane $$\lambda x + \mu y + \nu z = 0,$$

is (2) ..... $$\frac{\frac{d\mathrm{F}}{dl}}{\lambda} = \frac{\frac{d\mathrm{F}}{dm}}{\mu} = \frac{\frac{d\mathrm{F}}{dn}}{\nu}.$$

To obtain, in a form similar to (1), the condition for the perpendicularity of two planes, we introduce the *contravariant* of F, *i.e.*, the function

$$\Phi(\xi, \eta, \zeta) = \xi^2 \sin^2 \mathrm{X} + \eta^2 \sin^2 \mathrm{Y} + \zeta^2 \sin^2 \mathrm{Z} + 2\eta\zeta(\cos \mathrm{Y} \cos \mathrm{Z} - \cos \mathrm{X}) + 2\zeta\xi(\cos \mathrm{Z} \cos \mathrm{X} - \cos \mathrm{Y}) + 2\xi\eta(\cos \mathrm{X} \cos \mathrm{Y} - \cos \mathrm{Z}),$$

or, as it may be written, if X_1, Y_1, Z_1 are the angles between the normals to the coordinate planes,

$$\Phi(\xi, \eta, \zeta) = \xi^2 \sin^2 \mathrm{X} + \eta^2 \sin^2 \mathrm{Y} + \zeta^2 \sin^2 \mathrm{Z} + 2\eta\zeta \sin \mathrm{Y} \sin \mathrm{Z} \cos \mathrm{X}_1 + 2\zeta\xi \sin \mathrm{Z} \sin \mathrm{X} \cos \mathrm{Y}_1 + 2\xi\eta \sin \mathrm{X} \sin \mathrm{Y} \cos \mathrm{Z}_1.$$

The condition for the perpendicularity of the two planes

$$\lambda x + \mu y + \nu z = 0,$$
$$\lambda' x + \mu' y + \nu' z = 0,$$

may now be expressed in either of the two equivalent forms

(3)..... $$\begin{cases} \lambda' \dfrac{d\Phi}{d\lambda} + \mu' \dfrac{d\Phi}{d\mu} + \nu' \dfrac{d\Phi}{d\nu} = 0, \\ \lambda \dfrac{d\Phi'}{d\lambda'} + \mu \dfrac{d\Phi'}{d\mu'} + \nu \dfrac{d\Phi'}{d\nu'} = 0; \end{cases}$$

and the condition (2) for the perpendicularity of a plane and a line may be expressed in a second, and equally convenient form,

(4) ..... $$\frac{\frac{d\Phi}{d\lambda}}{l} = \frac{\frac{d\Phi}{d\mu}}{m} = \frac{\frac{d\Phi}{d\nu}}{n}.$$

The formulæ (1), (2), (3), (4) are well known, and are only given here for convenience of reference; their demonstration by the ordinary methods of analytical geometry presents no difficulty. We may, however, call attention to the following points:—

(i.) The equation $\mathrm{F}(xyz) = 0$ represents a sphere of evanescent

radius having its centre at the origin; or, which is the same thing, a certain imaginary cone having its vertex at the origin, and asymptotic to any sphere concentric with the origin. Such an imaginary sphere-cone possesses the characteristic properties, that (α) any two lines, (β) any line and plane, (γ) any two planes, which are harmonically conjugate with regard to it, are at right angles to one another. The conditions (1) and (2) are the analytical expression of the properties (α) and (β).

(ii.) Again, the tangential equation of the imaginary sphere-cone, *i.e.*, the condition that the plane

$$\xi x + \eta y + \zeta z = 0$$

should be one of its tangent planes, is

$$\Phi(\xi, \eta, \zeta) = 0;$$

and the conditions of perpendicularity (3) and (4) are the analytical expressions, by means of the tangential equation Φ, of the properties (γ) and (β).

(iii.) As the formula $F(x, y, z)$ expresses the square of any rectilineal segment in terms of its oblique projections upon the three coordinate axes; so the formula $\Phi(\xi, \eta, \zeta)$ expresses the square of any plane area in terms of its oblique projections $\xi \sin X$, $\eta \sin Y$, $\zeta \sin Z$, upon the three coordinate planes. The imaginary sphere-cone possesses, it will be noticed, the two paradoxical properties, that the square of any segment lying on one of its lines, and the square of any area lying on one of its tangent planes, are each of them equal to zero.

4. *Conditions of Perpendicularity in a Parallelepipedal System.*

Adopting the notation of the classical treatise of Professor Miller, we designate by a, b, c the parameters of a parallelepipedal system; we thus have, for the square of the distance between any two points of the system, the expression

$$\begin{aligned} f(x, y, z) &= F(ax, by, cz) \\ &= a^2x^2 + b^2y^2 + c^2z^2 + 2bcyz \cos X + 2cazx \cos Y + 2abxy \cos Z, \end{aligned}$$

where $x\,y\,z$ now denote integral numbers. Again, if

$$\begin{aligned} \phi(\xi, \eta, \zeta) &= \Phi(bc\xi, ca\eta, ab\zeta) \\ &= b^2c^2\xi^2 \sin^2 X + c^2a^2\eta^2 \sin^2 Y + a^2b^2\zeta^2 \sin^2 Z \\ &\quad + 2a^2bc\eta\zeta \sin Y \sin Z \cos X_1 + 2b^2ca\zeta\xi \sin Z \sin X \cos Y_1 \\ &\quad + 2c^2ab\xi\eta \sin X \sin Y \cos Z_1, \end{aligned}$$

the form ϕ, which is the contravariant of f, characterizes (in the manner indicated by Gauss) a new parallelepipedal system (the polar system of

Bravais) in which every line is perpendicular to a plane of the given system, and in which the parameter corresponding to any line is the elementary parallelogram in the plane to which the line is perpendicular.

Writing, for brevity,

$$f = Ax^2+By^2+Cz^2+2A'yz+2B'xz+2C'xy,$$
$$\phi = A_1\xi^2+B_1\eta^2+C_1\zeta^2+2A_1'\eta\zeta+2B_1'\zeta\xi+2C_1'\xi\eta,$$

(so that $A = a^2$, ..., $A' = bc\cos X$, ... $A_1 = b^2c^2\sin^2 X$, ...
$A_1' = a^2bc \sin Y \sin Z \cos X_1$, ...)

and denoting the determinant

$$\begin{vmatrix} A, & C', & B' \\ C', & B, & A' \\ B', & A', & C' \end{vmatrix} = \begin{cases} a^2b^2c^2\,(1-\cos^2 X-\cos^2 Y-\cos^2 Z \\ \qquad +2\cos X\cos Y\cos Z) \end{cases}$$

by Δ, we have the well known relations

$$A_1 = BC-A'^2, \quad B_1 = CA-B'^2, \quad C_1 = AB-C'^2,$$
$$A_1' = B'C'-AA', \quad B_1' = C'A'-BB', \quad C_1' = A'B'-CC',$$

$$\begin{vmatrix} A_1, & C_1', & B_1' \\ C_1', & B_1, & A_1' \\ B_1', & A_1', & C_1 \end{vmatrix} = \Delta^2,$$

$$\Delta A = B_1C_1-A_1'^2, \quad \Delta B = C_1A_1-B_2'^2, \quad \Delta C = A_1B_1-C_1'^2,$$
$$\Delta A' = B_1'C_1'-A_1A_1', \quad \Delta B' = C_1'A_1'-B_1B_1', \quad \Delta C' = A_1'B_1'-C_1C_1';$$

which serve to show that, if we suppose $\Delta = 1$, or the volume of the elementary parallelepiped of the system f to be unity (a supposition which is admissible, because, if Δ be not unity, we may alter the parameters in the ratio of $\sqrt[6]{\Delta} : 1$), the relation between the two parallelepipedal systems is reciprocal; *i.e.*, the system f is derived from the system ϕ in the same way in which the system ϕ has been derived from the system f.

The five quantities upon which the nature of the parallelepipedal system ultimately depends are unquestionably the angles XYZ, and the ratios of the parameters *abc*. But the combinations of these quantities which enter into the conditions of perpendicularity and symmetry are precisely the six *covariant* coefficients

$$ABCA'B'C'$$

and the six *contravariant* coefficients

$$A_1B_1C_1A_1'B_1'C_1'.$$

Thus, if (5)..... $\begin{cases} \dfrac{x}{au} = \dfrac{y}{bv} = \dfrac{z}{cw} \\ \dfrac{x}{au_1} = \dfrac{y}{bv_1} = \dfrac{z}{cw_1} \end{cases}$

are any two lines of the system, the condition for their perpendicularity is

$$u_1\frac{df}{du}+v_1\frac{df}{dv}+w_1\frac{df}{dw}=0,$$

or

$$u\frac{df_1}{du_1}+v\frac{df_1}{dv_1}+w\frac{df_1}{dw_1}=0,$$

i. e., the condition that the lines (5) should be perpendicular is the same as the condition that the lines

$$(6)\ldots\ldots\begin{cases}\dfrac{x}{u}=\dfrac{y}{v}=\dfrac{z}{w}, & \dfrac{x}{u_1}=\dfrac{y}{v_1}=\dfrac{z}{w_1},\end{cases}$$

should be harmonically conjugate with regard to the cone $f=0$. Again, the condition that the two planes of the system

$$(7)\ldots\ldots\begin{cases}\dfrac{hx}{a}+\dfrac{ky}{b}+\dfrac{lz}{c}=0,\\[2ex] \dfrac{h_1x}{a}+\dfrac{k_1y}{b}+\dfrac{l_1z}{c}=0,\end{cases}$$

should be perpendicular to one another is the same as the condition that the planes

$$(8)\ldots\ldots\begin{cases}hx+ky+lz=0,\\ h_1x+k_1y+l_1z=0,\end{cases}$$

should be harmonically conjugate with regard to the same cone $f=0$, of which the tangential equation is $\phi=0$; viz., this condition is

$$h_1\frac{d\phi}{dh}+k_1\frac{d\phi}{dk}+l_1\frac{d\phi}{dl}=0,$$

or

$$h\frac{d\phi_1}{dh_1}+k\frac{d\phi_1}{dk_1}+l\frac{d\phi_1}{dl_1}=0.$$

And similarly the condition that one of the lines (5) should be perpendicular to one of the planes (7), is the same as the condition that the corresponding line (6) and plane (8) should be a polar line and polar plane with regard to the cone $f=0$ or $\phi=0$.

We shall, in what follows, adopt the mode of expressing the conditions of perpendicularity which has just been explained. By rational lines, planes, or cones, we shall understand lines, planes, or cones, of which the equations have integral numbers for their coefficients. We shall term the lines and planes (6) and (8) the rational lines and planes corresponding to the lines and planes of the system (5) and (7). Except in the case of a spherical symmetry, the cone $f=0$, or $\phi=0$, is not itself a rational cone: and thus (in all other cases) the problem, "to determine all the lines and planes which are perpendicular to one another in a given parallelepipedal system," is the same as the problem, "to

find all the rational lines and planes which are harmonically conjugate to one another with regard to a given irrational cone."

5. *Linear Relations connecting the Coefficients.*

Let the parallelepipedal system contain a pair of perpendicular lines (5); the condition of perpendicularity gives immediately the relation

$$(9)\ldots\ldots \begin{cases} Auu_1+Bvv_1+Cww_1+A'(vw_1+wv_1) \\ \quad +B'(wu_1+uw_1)+C'(uv_1+u_1v)=0. \end{cases}$$

Unless therefore the six covariant coefficients are connected by a linear homogeneous equation having integral coefficients, no two lines of the system can be perpendicular to one another; and, correlatively, unless the six contravariant coefficients are connected by a similar relation, no two planes of the system can be perpendicular to one another. But the existence of such a relation connecting the six coefficients (or the six contravariant coefficients), though a necessary condition, is not a sufficient condition for the existence in the system of a pair of perpendicular lines (or planes). We proceed, therefore, to examine more closely the circumstances which present themselves, when the coefficients are connected by one, two, three, four, or five relations. By a relation connecting the coefficients we understand a linear homogeneous equation of the type

$$(10)\ldots\ldots pA+qB+rC+2p'A'+2q'B'+2r'C'=0,$$

in which $pqrp'q'r'$ are integral numbers; in connection with such a relation we shall have to consider the quadratic form

$$\psi'=p\xi^2+q\eta^2+r\zeta^2+2p'\eta\zeta+2q'\zeta\xi+2r'\xi\eta,$$

and its reciprocal form

$$\begin{aligned}\Psi=&(p'^2-qr)\,x^2+(q'^2-rp)\,y^2\\&+(r'^2-pq)\,z^2+2\,(pp'-q'r')\,yz\\&+2\,(qq'-r'p')\,zx+2\,(rr'-p'q')\,xy;\end{aligned}$$

these we shall term the quadratic form and the reciprocal quadratic form appertaining to the given relation.

For brevity, we shall attend only to the cases in which given relations exist between the six covariant coefficients ABC ...; the cases, in which given relations exist between the six contravariant coefficients, are simply the correlatives of these; conditions for the perpendicularity of two planes answering to conditions for the perpendicularity of two lines, and every condition of symmetry admitting of a two-fold expression according as we employ the covariant, or the contravariant coefficients.

It is remarkable that, in every case, the conditions of perpendicularity and symmetry depend solely on the coefficients of the linear relations connecting the crystallographic coefficients; so that two parallelepipedal systems in which the crystallographic coefficients have different ratios, but satisfy the same linear relations, would resemble one another exactly in respect of symmetry and perpendicularity.

6. *Case of one Linear Relation between the Coefficients.*

Here we have the theorem:—

"The system contains a single pair of perpendicular lines, or contains no such pair whatever, according as the reciprocal form appertaining to the given relation is, or is not, a perfect square."

For (i.) if the system contain a pair of perpendicular lines, the coefficients are connected by a relation of the type (9); and the reciprocal form appertaining to this relation is the perfect square

$$\tfrac{1}{4} \times \begin{vmatrix} x, & y, & z \\ u, & v, & w \\ u_1, & v_1, & w_1 \end{vmatrix}^2$$

Conversely, (ii.) if the coefficients are connected by a relation of the type (10), which satisfies the condition

$$\Psi = \square,$$

the system contains a single pair of perpendicular lines. Let

$$\Psi = (\alpha x + \beta y + \gamma z)^2,$$

and consider the matrix

$$\begin{vmatrix} p, & r'-\gamma, & q'+\beta \\ r'+\gamma, & q, & p'-\alpha \\ q'-\beta, & p'+\alpha, & r \end{vmatrix}$$

in which it is easily proved that every first minor is equal to zero, and in which we may suppose that the nine elements are freed by division from any common divisor which they may possess. If in this matrix we represent by uvw the greatest common divisors of the elements in the first, second, and third rows respectively, we find immediately

$$p = uu_1, \quad q = vv_1, \quad r = ww_1,$$
$$2p' = vw_1 + wv_1, \quad 2q' = wu_1 + uw_1, \quad 2r' = uv_1 + vu_1;$$

whence it follows that the equation (10) assumes the form of the equation (9), and that the two lines

$$\frac{x}{au} = \frac{y}{bv} = \frac{z}{cw},$$

$$\frac{x}{au_1} = \frac{\eta}{bv_1} = \frac{z}{cw_1},$$

are perpendicular to one another.

For the condition that Ψ must be a perfect square, we may, if we please, substitute the condition that the form ψ appertaining to the given relation should resolve itself into two linear factors having integral coefficients. If these factors are

$$u\xi+v\eta+w\zeta,$$
$$u_1\xi+v_1\eta+w_1\zeta,$$

the lines (5) are two perpendicular lines.

It is important to observe that the reciprocal form Ψ cannot be identically equal to zero. If it were so, the form ψ would itself be a perfect square, and the cone $f=0$, which is certainly imaginary, would contain a real line. Again, when we say that Ψ is to be a perfect square, we understand that Ψ itself is to be the square of a linear function of xyz (and not merely a multiple of such a square by a number which is not itself a square). Thus the condition $\Psi = \square$ may be replaced by the two conditions, (1) that the discriminant of ψ is to be zero, (2) that the greatest common divisor of the first minors of this discriminant is to be a perfect square; and of these the second may, if we please, be replaced by the condition that, of the three principal minors, one, which is not zero, is to be a perfect square (these three minors cannot all be zero, and if one of them is a perfect square different from zero, the other two are either zero or perfect squares).

7. Since the form f is positive and definite, the coefficients A, B, C, A′, B′, C′ are subject to certain inequalities. But, subject to these, they may be any quantities whatever, rational or irrational. Hence no relation of the form (10) can in general subsist between them; or, which is the same thing, a parallelepipedal system cannot, in the most general case, contain a single right angle. The problem, "Given any number of irrational quantities to determine the rational linear relations (if any) which subsist between them," presents great difficulties, and, in the present state of indeterminate analysis, is perhaps insoluble. We observe, however, that if $m+n$ quantities $Y_1, Y_2, \ldots Y_{m+n}$ are connected by m independent equations of the form

$$g_{i,1}Y_1+g_{i,2}Y_2+\ldots\ldots+g_{i,m+n}Y_{m+n}=0,$$

where $i=1, 2, 3, \ldots m$, and the numbers g are integral, we can always (and indeed, unless $n=1$, in an infinite number of ways) assign n quantities Z_1, Z_2, Z_n such that the $m+n$ quantities Y can be expressed in terms of these by means of a system of equations of the type

$$Y^j = h_{j,1}Z_1+h_{j,2}Z_2+\ldots\ldots+h_{j,n}Z_n,$$

where the numbers h are integral, and $j=1, 2, 3, \ldots m+n$.*

* See the memoir to which we have already referred, Phil. Trans., 1861, p. 299, *sq.*

It follows from this theorem that, if the coefficients are connected by one and only one relation, there is an equation of the type

$$f(xyz) = \omega_1\Sigma_1 + \omega_2\Sigma_2 + \ldots\ldots \omega_5\Sigma_5,$$

where $\Sigma_1, \Sigma_2, \ldots \Sigma_5$ are quadratic forms in x, y, z having integral coefficients, and $\omega_1, \omega_2, \ldots \omega_5$ are irrational quantities between which no linear rational homogeneous equation can exist. In general, if $i+j=6$, and if there are i relations between the coefficients, we have an equation of the type $\qquad f = \omega_1'\Sigma_1 + \ldots\ldots \omega_j\Sigma_j.$

The rational cones of which the equations are

$$\Sigma_1 = 0, \quad \Sigma_2 = 0, \quad \ldots\ldots \quad \Sigma_j = 0,$$

and again the rational cones of which the tangential equations are

$$\psi_1(\xi\eta\zeta) = 0, \quad \psi_2(\xi\eta\zeta) = 0, \quad \ldots\ldots \quad \psi_i(\xi\eta\zeta) = 0,$$

determine two systems of contravariant cones; every cone of the system (Σ) circumscribing harmonically every cone of the system (ψ), and, *vice versâ*, every cone of the system (ψ) being harmonically inscribed in every cone of the system (Σ).*

We have seen that the existence of a pair of perpendicular lines in a parallelepipedal system implies the existence of a relation between the coefficients; but that not every such relation implies the existence of a pair of perpendicular lines. When the reciprocal form Ψ appertaining to a given relation is not a perfect square, we can only assert that the cone f harmonically circumscribes the cone ψ. But this statement seems to have no crystallographic meaning.

8. *Case of two linear relations between the coefficients.*

Let ψ_1, ψ_2; Ψ_1, Ψ_2 be the forms appertaining to the two given linear relations; and let us at first suppose that

$$\Psi_1 = \square, \quad \Psi_2 = \square;$$

so that the system contains two pairs of perpendicular lines.

Three particular cases require especial notice.

(1) If the squares Ψ_1 and Ψ_2 are identical, or differ only by a numerical factor, so that, retaining the notation of Art. 6, we have the equations, $\qquad \frac{\alpha_1}{\alpha_2} = \frac{\beta_1}{\beta_2} = \frac{\gamma_1}{\gamma_2},$

the planes of the two right angles coincide; and every line of the system that lies in this plane has a line of the system at right angles to it in the same plane.

(2) The plane of one of the right angles may contain one of the rays of the other right angle.

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* Proceedings of the London Mathematical Society, Vol. ii., pp. 90, 91.

(3) The two right angles may have one ray in common; their common ray is then at right angles to the plane of the other two rays, and this plane becomes a plane of simple symmetry. The condition that this should happen is that ψ_1 and ψ_2 should have a linear factor in common; viz., if these two forms have a common linear factor, this factor is of necessity rational, the two reciprocal forms are perfect squares, and the common factor is

$$\begin{vmatrix} \xi, & \eta, & \zeta \\ \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \end{vmatrix}$$

Still retaining the hypothesis that Ψ_1 and Ψ_2 are perfect squares, but excluding the particular cases to which we have just referred, we observe that, besides the two pairs of perpendicular lines which we obtain immediately from the two given relations, the system contains one and only one other pair of perpendicular lines. This is a consequence of the elementary theorem that if AOB, COD are two right angles in different planes, the intersection of the planes AOD, BOC is at right angles to the intersection of the planes AOC, BOD; a theorem which is itself a particular case of a general property of two pairs of conjugate lines with regard to any cone (or conjugate points with regard to any conic), first given by Otto Hesse.

We now quit the hypothesis that Ψ_1 and Ψ_2 are squares. When only one linear relation subsists between the coefficients, we are certain that, if the reciprocal form appertaining to it is not a perfect square, the parallelepipedal system cannot contain a single pair of perpendicular lines. But, when we have two such relations, we have still to inquire whether, by combining them linearly with one another, we cannot obtain a new relation which shall satisfy the condition

$$\Psi = \square.$$

Let θ, θ_1, θ_2 be the three roots of the discriminantal cubic of $\psi_1 + \theta\psi_2$. If these roots are irrational, the system contains not a single pair of perpendicular lines. If one of them, for example θ, is rational, we have still to examine whether the factors of $\psi_1 + \theta\psi_2$ are rational; if they are, we have a pair of perpendicular lines. If all the three roots are rational, we have to examine the factors of the three forms $\psi_1 + \theta\psi_2$, $\psi_1 + \theta_1\psi_2$, $\psi_1 + \theta_2\psi_2$; according as these factors are or are not rational (if the factors of two of the forms are rational, the factors of the third are also rational), we obtain one or three pairs of perpendicular lines, or no pair at all of such lines.

If two of the roots of the discriminating cubic are equal, the cones ψ_1 and ψ_2 touch one another; if the three roots are all equal, these cones osculate (the cases of double contact and of super-osculation cannot

present themselves, as they would imply the existence of a real line on the cone $f = 0$). When two of the roots are equal, we have either the special case (2) considered above, or we have one and only one pair of perpendicular lines; when the three roots are all equal, we have a single pair of perpendicular lines.

Lastly, the coefficients of the discriminating cubic may all vanish. If this happens, either (α) Ψ_1 and Ψ_2 differ (if at all) only by a numerical factor, or (β) ψ_1 and ψ_2 have a common linear factor. The case (α) is the particular case (1) to which we have already referred. For each of the equations $\psi_1 = 0$, $\psi_2 = 0$ represents a pair of lines in the plane $\surd\Psi_1 = 0$ or $\surd\Psi_2 = 0$; then two pairs of lines may themselves be irrational, but, even if they are, every rational line lying in their plane has a rational conjugate in the involution determined by them, and the pairs of this involution are conjugate pairs with regard to the cone $f = 0$.

The case (β), when ψ_1 and ψ_2 have a common linear factor, is the case of simple symmetry (the particular case (3) already mentioned).

We may therefore enunciate the theorem :—

"The conditions that a parallelepipedal system should possess a simple symmetry are—

"(a) That the coefficients should be connected by two linear relations;

"(b) That the two quadratic forms appertaining to these relations should have a linear factor in common."

For the condition (b) we may substitute the following :—

"The four invariants of ψ_1 and ψ_2 must vanish; the first minors of the discriminants of ψ_1 and ψ_2 not being proportional to one another."

This is only an explicit statement of the conditions which are necessary and sufficient to ensure the presence of a common factor in ψ_1 and ψ_2.

9. *Case of three linear relations between the coefficients.*

If the parallelepipedal system includes three pairs of lines at right angles to one another (these pairs being *asyzygetic*, *i.e.*, such that the existence of one of them is not a necessary consequence, in the manner already explained, of the existence of the other two), the six coefficients are connected by three independent linear relations of the type (10), and the reciprocal forms appertaining to these relations are perfect squares. Conversely, when these conditions are satisfied, the system contains not merely three pairs of lines at right angles to one another, but (as we shall presently see) an infinite number of such pairs; indeed, this will happen even if only one of the three reciprocal forms is a perfect square. And, even when none of these forms is a perfect square, the system *may* contain rectangular pairs of lines, because (as in the

last case) it may be possible to combine linearly the given relations in such a manner as to obtain a new relation satisfying the condition

$$\Psi = \square.$$

Such a combination, as we shall now prove, is, or is not, possible, according as the indeterminate equation

$$C = \begin{vmatrix} \frac{d\psi_1}{d\xi}, & \frac{d\psi_1}{d\eta}, & \frac{d\psi_1}{d\zeta} \\ \frac{d\psi_2}{d\xi}, & \frac{d\psi_2}{d\eta}, & \frac{d\psi_2}{d\zeta} \\ \frac{d\psi_3}{d\xi}, & \frac{d\psi_3}{d\eta}, & \frac{d\psi_3}{d\zeta} \end{vmatrix} = 0$$

does, or does not, admit of resolution in integral numbers.

In accordance with the theorem of Art. 7, we represent f by an equation of the type

$$f = \omega_1\Sigma_1 + \omega_2\Sigma_2 + \omega_3\Sigma_3;$$

and, since no linear relation can exist between ω_1, ω_2, ω_3, we infer that any pair of rational lines which are conjugate with regard to f are also conjugate with regard to Σ_1, Σ_2, and Σ_3; the problem before us, therefore, is, "To determine all the rational lines which are conjugate with regard to Σ_1, Σ_2, Σ_3."

With reference to this problem, we consider the system of contravariant cones ψ_1, ψ_2, ψ_3, and Σ_1, Σ_2, Σ_3; the equations $\psi = 0$ being equations in plane coordinates, and the equations $\Sigma = 0$ being equations in point coordinates. The Cayleyan cone of the system is the cone of the third class $C = 0$, of which the equation has been already given. This cone is the Jacobian of the cones ψ_1, ψ_2, ψ_3; the Jacobian of the cones Σ_1, Σ_2, Σ_3 is the Hessian cone of the system; its equation is

$$H = \begin{vmatrix} \frac{d\Sigma_1}{dx}, & \frac{d\Sigma_1}{dy}, & \frac{d\Sigma_1}{dz} \\ \frac{d\Sigma_2}{dx}, & \frac{d\Sigma_2}{dy}, & \frac{d\Sigma_2}{dz} \\ \frac{d\Sigma_3}{dx}, & \frac{d\Sigma_3}{dy}, & \frac{d\Sigma_3}{dz} \end{vmatrix} = 0.$$

The Cayleyan C is a combinantive covariant of the three forms ψ_1, ψ_2, ψ_3; the Hessian H is a combinantive contravariant of the same three forms; the developed expressions of C and H have been given by Dr. Salmon,* and it is unnecessary to repeat them here.

All the pairs of lines harmonically conjugate with regard to the system of cones (Σ) lie on the cubic cone $H=0$; and *vice versâ*, every

* Conic Sections, Ed. 5, pp. 346, 347.

line lying on this cubic cone has a corresponding line lying on the same cone which is its conjugate with regard to all the cones of (Σ); it only remains, therefore, to ascertain whether any of these pairs of conjugate lines are rational. The problem, "To determine whether a homogeneous cubic equation, containing three indeterminates, and having integral coefficients, does, or does not, admit of solution in integral numbers," is one of which, in its general form, no complete discussion has as yet been given. It is obvious, however, that, if the equation $H = 0$ admits of a single solution in integral numbers, it admits of an infinite number of such solutions. In general this is true for any homogeneous cubic equation containing three indeterminates, and having integral coefficients. For we may regard the cubic equation as defining a rational cubic cone; if there is a rational line appertaining to this cone, the successive *tangentials* of this rational line are themselves rational lines; and the plane meeting the cone in any two rational lines meets it in a third rational line. This mode of construction is subject to certain exceptions; if, for example, the given rational line is a line of inflexion, it is its own tangential, and gives rise to no second rational line. But in the case before us (when the cubic cone appears as the Hessian of a rational system of cones) the line conjugate to any given rational line of the Hessian cone is always rational, and distinct from the given line; and the plane joining the two lines meets the Hessian cone in a third line distinct from either of them, of which we may again take the conjugate line, and so on continually. The line conjugate to a given line of the Hessian cone is most readily found by taking the polar planes of the given line with regard to the cones (Σ); these polar planes are all rational, and intersect in a rational line, which is the line required. We may also note—(1) That if AA′, BB′ are two pairs of rational conjugate lines, the lines (AB′, A′B), (AB, A′B′) are themselves a pair of rational conjugate lines; (2) that if C is any rational line of the Hessian, the planes CA and CA′ meet the Hessian again in a pair of rational conjugate lines; (3) that A, A′ have the same rational tangential D, which is the conjugate of the line in which the plane AA′ meets the Hessian again.

From what precedes, it appears that, if the equation $H = 0$ admits of a single solution in integral numbers, the parallelepipedal system contains an infinite number of pairs of perpendicular lines all lying on the cubic cone $H(xa, yb, zc) = 0$. And the same thing is true (as has been already said) if the equation $C = 0$ admits of a single solution in integral numbers; for the two equations $C = 0$ and $H = 0$ are simultaneously resoluble or irresoluble; or, which is the same thing, if the Cayleyan cone $C = 0$ has rational tangent planes, the Hessian cone $H = 0$ has rational lines, and *vice versâ*; this, indeed, is evident, because the plane containing any two conjugate lines of the Hessian is a tangent

plane of the Cayleyan; and the intersection of any two conjugate planes tangent to the Cayleyan is a line of the Hessian.

10. *Three Relations between the Coefficients—Conditions of Symmetry.*

The system may have a simple symmetry, or an ellipsoidal symmetry, or none at all. But it cannot have a spheroidal or a spheric symmetry.

If there is a simple symmetry, the cones (Σ) must have in common a rational polar line and polar plane; but must not have in common a rational self-conjugate system of diametral planes. On the other hand, when there is an ellipsoidal symmetry, the cones (Σ) must have such a system of diametral planes.

(A.) *Conditions that the cones of the system* (Σ) *should have in common a single polar line and polar plane* [the line and plane are necessarily rational].

We take for the axis of z the polar line, the polar plane for the plane of xy, and for the axes of x and y the lines in which this polar plane is intersected by the tangent planes drawn from the axis of z to touch any one of the cones (ψ). (Except in a special case, which we shall notice presently, these cones all touch the same pair of planes intersecting in the axis of z.) We may then take for the representatives of the system (Σ) the three loci

$$x^2 = 0, \quad y^2 = 0, \quad z^2 - 2\lambda xy = 0;$$

and for the representatives of the system (ψ) the three envelopes

$$\zeta\eta = 0, \quad \zeta\xi = 0, \quad \lambda\zeta^2 + 2\xi\eta = 0;$$

λ being a constant different from zero. Thus the Hessian and the Cayleyan respectively become

$$xyz \quad \text{and} \quad \zeta(2\xi\eta - \lambda\zeta^2),$$

the former resolving itself into a product of three linear factors, the latter into a product of a linear by a quadratic factor. The axis of z is conjugate (with regard to the cones Σ) to every line in the plane xy; every line in either of the planes xz or yz is conjugate to a line in the other of those two planes; and the planes containing these pairs of conjugate lines envelope the Cayleyan cone $2\xi\eta - \lambda\zeta^2 = 0$.

The conditions that a ternary cubic should resolve itself into three linear factors is that the Hessian of the form should coincide with the form itself; and the condition that a ternary cubic should resolve itself into a linear and a quadratic factor is that the evectants of its two principal invariants should coincide.*

* Dr. Salmon's Higher Plane Curves, pp. 190 and 202, *sqq.*

The conditions, therefore, for a simple symmetry are that the Hessian of H should coincide with H, and that the two evectants of C should coincide with one another; but that the Hessian of C should not coincide with C.

The rational line corresponding to the axis of symmetry is the line, and the only line, common to all the quadratic cones represented by the first minors of the Hessian; the linear factor of C represents it tangentially. The rational plane corresponding to the plane of symmetry is the polar plane of this rational line with regard to the quadratic Cayleyan cone. One of the linear factors of the Hessian represents this polar plane, and is certainly rational; the other two factors (x and y) may be irrational, or even imaginary. When they are rational, there exist in the parallelepipedal system two planes, not at right angles to one another, intersecting in the axis of symmetry, and each containing an infinite number of lines of the system at right angles to one another.

These two planes may coincide; if they do, the plane of coincidence is certainly rational. The special case at which we thus arrive has been excluded from the preceding discussion; for we have supposed that the cones ψ all touch two distinct tangent planes intersecting in the common polar line of the cones Σ. But the cones ψ may instead touch two coincident planes; *i. e.*, they may all pass through the polar line, and touch a fixed plane along that line. If we take this plane for the plane of yz, and any other plane through the polar line for the plane of xz, we may represent the systems (Σ) and (ψ) by the loci

$$x^2 = 0, \quad xy = 0, \qquad y^2 + \lambda z^2 = 0,$$

and the envelopes

$$\eta\zeta = 0, \quad \xi\zeta = 0, \quad -\lambda\eta^2 + \zeta^2 = 0.$$

The Hessian and the Cayleyan are respectively

$$x^2 z \quad \text{and} \quad \zeta(\zeta^2 + \lambda\eta^2),$$

the former being a product of a linear by a square factor; the latter being a product of three linear factors, of which two are certainly imaginary, because, if λ were negative, the cones Σ would have two real lines in common, which would consequently appertain to the cone f. The conditions for this special case of simple symmetry, which is characterised by the presence of a single plane passing through the axis of symmetry, and containing an infinite number of pairs of perpendicular lines, are that the Hessian of H should vanish identically, and that the Hessian of C should coincide with C.

(B.) *Conditions that the system* (Σ) *should have in common a self-conjugate system of diametral planes, of which one at least is rational.*

The cones (ψ) must have three tangent planes in common (viz.,

the three self-conjugate planes of the system Σ). Thus the Cayleyan cone resolves itself into three lines (the three lines of intersection of the diametral planes), and the Hessian resolves itself into those three diametral planes themselves.

For this the necessary and sufficient conditions are that the Hessian of the Hessian should coincide with the Hessian, and the Hessian of the Cayleyan with the Cayleyan.

The three linear factors of the Cayleyan (or the Hessian) may, however, be all irrational, or one of them only may be rational, or they may all three be rational; and accordingly there may be either no symmetry at all, or a simple symmetry, or an ellipsoidal symmetry.

To distinguish between these three cases it is sufficient to examine the discriminantal cubic of any two of the cones Σ, or of any two of the cones ψ; for it is not difficult to verify that the linear factors of the Cayleyan, or Hessian, are rational or irrational according as the roots of any one of these equations are rational or irrational.

11. *Three Relations between the Coefficients—Special Cases.*

Some special cases, in which there is no symmetry, have (nevertheless) a certain interest. Of these we may briefly mention two.

(1) The cones ψ have one, and only one, tangent plane A in common. The Hessian in this case consists of A, and of a quadratic cone which is the *reciprocal* cone of A with regard to the cones of Σ. Thus, in the parallelepipedal system, there is a cone and plane such that every line lying in the plane has a line at right angles to it lying in the cone, and *vice versâ*; the lines of intersection of the plane and cone are at right angles, and *may* be lines of the system.

(2) The cones ψ have a polar line and polar plane in common.

The result is as follows:—The parallelepipedal system has an infinite number of pairs of lines at right angles to one another, all lying in the same plane; and it *may* have a second set of such pairs lying on the surface of a quadratic cone, the plane of each such pair passing through the polar line of the first-named plane with regard to the cone.

12. *Case of Four Linear Relations between the Coefficients.*

In this case the form f may be expressed by an equation of the type

$$f = \omega_1\Sigma_1 + \omega_2\Sigma_2,$$

where the ratio of ω_1 and ω_2 is irrational, and where the cones Σ_1, Σ_2 cannot have any real line in common.

Every rational line has a rational line conjugate to it with regard to each of the cones $\Sigma_1=0$, $\Sigma_2=0$, and consequently with regard to the cone $f=0$. Hence every line of the parallelepipedal system has a line at right angles to it; and this distribution of pairs of perpendicular lines may exist without the presence of any symmetry whatever.

Conditions of Symmetry.

The symmetry (if any) may be simple, or ellipsoidal, or spheroidal, but cannot be spherical.

There is a simple symmetry when the discriminantal cubic of the system (Σ_1, Σ_2) has one rational root, and an ellipsoidal symmetry when this cubic has three rational and unequal roots. When two of the roots are equal, the cones Σ_1 and Σ_2 have an imaginary double contact, and the axis and plane of the double contact are the axis and equatorial plane of a spheroidal symmetry.

If we represent by S the harmonic covariant of Σ_1 and Σ_2, the equation

$$J = \begin{vmatrix} \frac{d\mathrm{S}}{dx}, & \frac{d\mathrm{S}}{dy}, & \frac{d\mathrm{S}}{dz} \\ \frac{d\Sigma_1}{dx}, & \frac{d\Sigma_1}{dy}, & \frac{d\Sigma_1}{dz} \\ \frac{d\Sigma_2}{dx}, & \frac{d\Sigma_2}{dy}, & \frac{d\Sigma_2}{dz} \end{vmatrix} = 0$$

represents the three diametral planes which are self-conjugate with regard to the cones Σ_1 and Σ_2; and its left-hand member J vanishes identically when Σ_1 and Σ_2 have double contact.

The Jacobian J is a combinantive covariant of Σ_1 and Σ_2; its coefficients therefore involve only the determinants (of the second order) which can be formed with the coefficients of Σ_1 and Σ_2. But these determinants* are proportional to the reciprocal determinants (of the fourth order) formed with the coefficients of $\psi_1, \psi_2, \psi_3, \psi_4$; *i. e.*, the coefficients of the Jacobian can be expressed in terms of these reciprocal determinants. To avoid introducing a new notation, we abstain from giving the developed expression of J, which indeed is not required here. It will be observed, however, that, in the case which we are now considering, the conditions for the various kinds of symmetry, and the determination of the planes of symmetry, depend on a single ternary cubic form (resoluble into three linear factors) which is a combinantive contravariant of the four forms ψ. (See the concluding remark in Art. 5.)

13. *Case of Five Linear Relations between the Coefficients.*

In this case the ratios of the coefficients are themselves evidently rational; and the parallelepipedal system has a spherical symmetry.

For, the coefficients of f being rational (after division, if necessary, by a common irrational factor), every rational line has a rational polar plane; *i. e.*, every line of the parallelepipedal system has a plane of the system at right angles to it.

* See Phil. Trans., 1861, p. 301.

And, conversely, if a parallelepipedal system has a spherical symmetry, the ratios of the coefficients are rational; for, if we consider any three lines of the system at right angles to one another, we obtain three independent relations (see Art. 5) between the coefficients; and if we consider any line of the system (not lying in the same plane with two of the first three), and the plane perpendicular to it, we obtain two more relations between the coefficients, independent of one another and of the former three; *i. e.*, the ratios of the coefficients are rational.*

On Viscous Fluids in Motion. *By* J. G. Butcher, Esq., B.A.

[*Read December 14th*, 1876.]

Section I.

On the Formation of the Equation of Motion.

1. The usual broad division of bodies is into solids and fluids. Solid bodies are defined as those which sustain a longitudinal pressure without the support of a lateral pressure: fluids are those which cannot. A further subdivision is found convenient. Fluids are either liquid or gaseous. If a small quantity of *liquid* be put in a closed vessel, it will only partially fill the vessel, and will have a bounding surface other than the walls of the vessel. A *gas*, on the other hand, expands so as to fill the whole of the containing vessel.

Convenient as these distinctions are, it would seem that no rigid line can be drawn between the different states of body. Most bodies are capable of existing in all three of the above-mentioned states, and may be made to pass from one state to another by a series of gradations of more or less abruptness. We might expect therefore that, whatever hypothesis we adopt as to the constitution of bodies, the general equations to express their motion will, *by a proper adaptation of constants*, be applicable to any of these three states, or to any intermediate state, indifferently.

2. This view of the gradual alteration of state is borne out by what we know of the probable molecular constitution of bodies.

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* The question of the rationality or irrationality of the ratios of the crystallographic coefficients had already attracted the attention of Gauss, who, as appears from his Life ["Gauss: zum Gedachtniss;" by W. Sartorius von Waltershausen; Leipzig, 1856], had in the year 1831 devoted himself with great ardour to the study of crystallography. See the concluding paragraphs of the Review of Seeber's "Untersuchungen" already cited.

It seems probable that a solid body is composed of molecules in continual agitation; but the internal forces are so strong that a molecule never gets separated from its immediate surroundings, but oscillates continually about some mean position of equilibrium. As heat is given to the body, the molecular agitation increases, and some molecules get separated from their surroundings, while some remain stable. In this state the body may be called first a soft solid, then a viscous fluid. The viscosity of the fluid is determined by the ratio of the average number of molecules which remain stable to the number of those which get separated. As yet, however, the velocity of a molecule is not sufficient to carry it away from the action of its neighbours beyond the bounding surface of the liquid. By an increased application of heat, the molecules at the bounding surface acquire such velocities as to enable them to break loose from the rest of the body; and there ceases to be an accurately defined bounding surface. This stage marks the transition of a liquid into a gas. The effect goes on increasing (as has been experimentally verified in certain cases) till the body reaches the completely gaseous stage, which is characterised by an unlimited power of diffusion.

3. The most intractable phenomena to explain are those relating to what has been termed "*Elastische Nachwirkung.*"* To account for these, we must frame our molecular hypothesis somewhat as follows. We regard a body as composed of small groups of molecules, differing from one another as regards their states and properties. These groups may be arranged into classes, in such a way that for each class there is a certain amount of strain, which may take place without preventing the group from recovering its primitive configuration when stress is removed. This definite amount of strain for each class may be called the "limit of elasticity" for that class.

Consider now a body composed of a single class of groups. If the limit of elasticity for this class have a certain finite value, then, for all strains below this particular value, every group will return to its original configuration, and the body behaves as an *elastic solid.*"† The potential energy is here a function of the strain alone; and as much work is done by the body in returning to its primitive state, as was done upon it in straining it. If in some parts of the body the strain exceed the "limit of elasticity" (still regarding all the groups as

* All the fundamental ideas in what immediately follows are due to Prof. Clerk Maxwell; though he is not, of course, responsible for every statement made.

† Throughout this paper I use the term "elastic" as having reference to a resistance against change of shape, *not* a resistance against change of volume. In the latter sense of the term, air and many other media, which I classify here as *viscous*, are justly termed *elastic*.

belonging to a single class), continuous displacement will take place in those parts; the body may be termed an *elastico-plastic* solid, and no longer forms a conservative system.

Consider next a body composed of a single class of molecules, for which the limit of elasticity is zero. It is this feature that characterises the state of fluidity. In this case, the molecular agitation is such, that, however small the strain (and even when it is zero), a certain proportion of groups are thrown out of their stable configurations every unit of time, pass through an unstable position, and re-unite in other configurations depending on the new state of the body, and such that the potential energy of the system is less than it was. The smaller this proportion of disturbed molecules, the more viscous is the fluid.

Thus we may distinguish two main classes of bodies; viz., those in which the limit of elasticity is not, and those in which it is, zero. In the former case, when change of shape has been produced by the application of stress, we observe a gradual creeping back toward the original state as soon as the stress is removed, provided the stress be not too great: in the latter case, we observe a gradual yielding to the continued action of stress, and no tendency to return when stress is removed.

4. Taking the latter class of bodies, viz. *fluid* bodies, we may suppose that, for a given fluid at a given temperature, the proportion of groups which have remained stable and which have been thrown out respectively during a small time δt, remains constant, at any rate for slow motions. Thus, let m be the number of groups in an element which are found to be stable at the end of the time δt, n be the number of groups which are found to have taken up new configurations at the end of that time. At the end of the first interval δt, a proportion $\frac{m}{m+n}$ of the original groups remain in their original configurations of stability; at the end of a second interval δt, $\left(\frac{m}{m+n}\right)^2$ of the original groups remain unaltered; and at the end of the r^{th} interval, a proportion $\left(\frac{m}{m+n}\right)^r$ of the original groups remain in their primitive configuration. Thus, the longer the strain lasts, the fewer of the original groups remain undecomposed.

In cases of fluid *motion*, we must take account of the fact that the yielding to the continued action of stress is gradual, not instantaneous.

In cases of fluid *equilibrium*, the strain may be supposed to have lasted for such a length of time that the number of continually stable groups may be practically neglected; and the fluid finally takes up the same position that it would have done had all the groups at once given up their primitive configurations; that is, had the yielding to the action

of stress been instantaneous. It must be observed, as before stated, that in no *fluid* is there any tendency to recover its shape.

In a perfect fluid the action of stress produces an instantaneous yielding; that is, every group of molecules becomes immediately disorganised, and a rearrangement of all the groups takes place, so as to meet the requirements of the deforming stress.

5. Suppose, next, a body which is composed of several different classes of molecules; viz, classes which become decomposed under strains x_1, x_2, &c. respectively, where x_1, x_2, &c. have certain finite values; and also of the class in which the "limit of elasticity" is zero. We may now classify the various states of body which are mainly distinguishable.

Call those classes of groups of molecules generally, for which the limit of elasticity is not zero, A groups; and that class for which the limit of elasticity is zero, the B groups.

I. *When a body after a long-continued strain returns to its primitive configuration, it is called elastic.*

In this case the B groups are sensibly absent; but we must have the conditions satisfied, that all the strains shall be less than any of the quantities x_1, x_2. ... This condition means that there are *limits to perfect elasticity*.

II. *When the body after strain returns only partially.*

Here we must distinguish two cases—

(1) When it returns *partially after being strained for a long time*, and *wholly* (or sensibly so) *after a short strain.*

These phenomena may be observed even for the case of an indefinitely small strain.

In this case both A and B groups must be present, but the groups B must be present in relatively small numbers. The body may be termed *elastico-viscous*, and we have the condition that the strain must nowhere exceed x_1, x_2, ...

According to this theory, these phenomena could not take place in cases of strains some of which exceed some of the quantities x_1, x_2, ...

(2) When it returns *only partially after being strained for a short time* (and, *a fortiori*, for a long time).

(*a*) When this is true in the case of an indefinitely small strain, groups A and B must be present, but group A in comparatively small numbers. Here some of the groups B exhibit complete disorganisation, but none of the groups A. The body is then of the class before called *elastico-viscous*, the strain being everywhere less than x_1, x_2, ...

(*b*) The phenomena may also be observed in the case in which some of the strains exceed some of the quantities x_1, x_2, ...

This is the case with most metals. Here the body may be called *elastico-plastico-viscous*, when both A and B groups are present, and all the B and some of the A groups break down; or *elastico-plastic*, when only A groups are present, and some of these A groups break down.

III. *When the body does not return at all.*

(1) If this is observed *after a long strain, and not after a short strain*, then

(*a*) If the strain be only indefinitely small, the B groups alone must be sensibly present. The body is then termed *viscous*.

(*b*) If the phenomena be observed only in the case of strains exceeding a certain value, then both A and B groups may be present, and the body may be termed *plastico-viscous*.

(2) If these phenomena are observed *even after a short strain*, then

(*a*) If it be observable in the case of indefinitely small strains, the B groups only are present, but in that peculiar state, before referred to, which constitutes *perfect fluidity*.

(*b*) Otherwise, it can only be observed in the case of strains which in some parts of the body at every instant exceed x_1, x_2, ... Here A groups only are present, and are continually breaking down; and the body is termed *plastic*. The peculiar feature that a body of this kind presents to the eye is, that no visible deformation appears to occur until the complete breaking-down process sets in. But, on the theory above explained, such a body permits deforming strains up to a certain value before it yields, though the resulting deformations may be, and indeed are, imperceptible to the eye.

6. Let us now revert to the case of *simple viscosity*, and make the assumption before alluded to—viz., that a certain proportion of groups of molecules per second are thrown out of their old configurations.

Let u, v, w be the resolved velocities at any point (xyz) of a fluid; the alterations in the time δt of the sides of an elementary rectangular parallelepiped, whose lengths are dx, dy, dz, are

$$\frac{du}{dx}dx\,\delta t, \quad \frac{dv}{dy}dy\,\delta t, \quad \frac{dw}{dz}dz\,\delta t,$$

neglecting infinitesimals of higher orders. Similarly the alterations in the angles are

$$\left(\frac{dv}{dz}+\frac{dw}{dy}\right)\delta t, \quad \&c.$$

Let the strains and stresses, computed as is usual in works on elasticity, be respectively s_1, s_2, s_3, U, V, W;
and N_1, N_2, N_3, T_1, T_2, T_3;

where $s_1+s_2+s_3=\theta=$ dilatation of the substance in the neighbourhood of the point (xyz) at a given time.

* If the body were an elastic solid, and α, β, γ were the component displacements of any point of the body infinitely near (x, y, z), we should have (see Thomson & Tait, *Nat. Phil.*, p. 513)

$$s_1 = \frac{d\alpha}{dx}, \quad \frac{ds_1}{dt} = \frac{du}{dx}, \quad s_2 = \&c.,$$

$$\frac{d\mathrm{U}}{dt} = \frac{dv}{dx} + \frac{dw}{dy}, \quad \frac{d\mathrm{V}}{dt} = \&c.$$

Now, if the substance were perfectly elastic, none of the groups of molecules contained in the element would be thrown out of their places, and we should have $\frac{ds_1}{dt} = \frac{du}{dx}$, &c. If the substance were a perfect fluid, all the groups of molecules in the element would be thrown out of their places every unit of time; and the strain could only consist of a uniform contraction or dilatation of the substance, in which state, if the stress were suddenly removed, no further readjustment of groups would take place.† The body being viscous, a certain proportion of groups of molecules in the element are thrown out of their places in every unit of time; and, in calculating the alteration of strain of the element in each unit of time, these groups must be considered as having passed into the state corresponding to perfect fluidity, viz., a state of strain consisting of uniform contraction or dilatation. At the beginning of each successive unit of time, all the groups in the element (including the groups thrown out in the preceding instant) are in just as good a state, speaking from the *viscous* point of view, as they were at the beginning of the previous instant: that is to say, the same proportion, and no more, of the entire number of groups in the element will be thrown out during any unit of time that were thrown out in the previous unit.

Now let m = number of groups in the element which are not thrown out per unit of time; and n = number of groups thrown out, that is, the number which pass from the state of strain (s_1, s_2, s_3, U, V, W) to the state ($\frac{1}{3}\theta$, $\frac{1}{3}\theta$, $\frac{1}{3}\theta$, 0, 0, 0).

Consider now the m groups at the end of time δt. In each of these groups, s_1 has changed by a gradual process to the value $s_1 + \frac{ds_1}{dt}\delta t$, just as if these groups were *elastic* groups.

In each of the n groups, s_1 has changed (it is immaterial to enquire

* For this important improvement on my original mode of procedure, I am indebted to Prof. Clerk Maxwell.

† The *assumption* involved here is that a uniform expansion (or contraction) of an element does not require any relative re-arrangement of groups among themselves, as the pressure remains equal in all directions.

how) to the value $\frac{1}{3}\theta$. Consequently, for the whole element,

$$\left\{\begin{array}{l}\text{Time variation of strain}\\ \text{in element in unit of}\\ \text{time}\end{array}\right\} = m\frac{ds_1}{dt} = \left\{\begin{array}{l}\text{Alteration of strain in}\\ \text{the } m \text{ groups which be-}\\ \text{have as though they were}\\ \text{elastic groups.}\end{array}\right\} + \left\{\begin{array}{l}\text{Alteration of strain in}\\ \text{the } n \text{ groups which ex-}\\ \text{hibit abrupt change}\end{array}\right\}$$

or
$$m\frac{ds_1}{dt} = m\frac{du}{dx} + n\left(\tfrac{1}{3}\theta - s_1\right),$$

i.e.,
$$\frac{ds_1}{dt} = \frac{du}{dx} - l\left(s_1 - \tfrac{1}{3}\theta\right),$$

where
$$l = \frac{n}{m} = \left\{\begin{array}{l}\text{constant for the temperature}\\ \text{under consideration}\end{array}\right\}.$$

Similarly
$$\frac{d\mathrm{U}}{dt} = \frac{dv}{dz} + \frac{dw}{dy} - l\mathrm{U},$$

with similar values for the other strains. This gives

$$s_1 = \frac{1}{l + \dfrac{d}{dt}}\left(\frac{du}{dx} + \frac{l}{3}\theta\right)$$

$$\mathrm{U} = \frac{1}{l + \dfrac{d}{dt}}\left(\frac{dv}{dz} + \frac{dw}{dy}\right).$$

Now if, as in Thomson & Tait, *Nat. Phil.*, p. 520, ν = rigidity of body, or resistance to distortion; κ = resistance to dilatation, we can with them (p. 526) easily compute the stresses as follows

$$\mathrm{N}_1 = \left(\kappa - \tfrac{2}{3}\nu\right)\theta + 2\nu s_1,$$
$$\mathrm{T}_1 = \nu\mathrm{U}.$$

Substituting these and similar values in the general equations of motion, viz.,

$$\frac{d\mathrm{N}_1}{dx} + \frac{d\mathrm{T}_3}{dy} + \frac{d\mathrm{T}_2}{dz} + \rho\left(\mathrm{X} - \frac{\mathrm{D}u}{\mathrm{D}t}\right) = 0,$$

and two similar equations, we get

$$\frac{1}{l + \dfrac{d}{dt}}\left\{\left(\kappa l + \left(\kappa + \tfrac{1}{3}\nu\right)\frac{d}{dt}\right)\frac{d\theta}{dx} - \nu\nabla^2 u\right\} + \rho\left(\mathrm{X} - \frac{\mathrm{D}u}{\mathrm{D}t}\right) = 0,$$

with two other equations, and

$$\frac{d\theta}{dt} = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz},$$

where, to conform to Quaternion notation, we have written

$$-\nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}.$$

When l is large compared with $\frac{d}{dt}$, that is, when the viscosity is small, and the motion not very variable (not as in sound), we may write the equation

$$\kappa \frac{d\theta}{dx} - \frac{\nu}{l}\nabla^2 u + \rho\left(\mathrm{X} - \frac{\mathrm{D}u}{\mathrm{D}t}\right) = 0,$$

or, putting $-p = \mathrm{C} + \kappa\theta$, and $\frac{\nu}{l} = \mu =$ coefficient of viscosity, the equation is

$$\frac{dp}{dx} + \mu\nabla^2 u = \rho\left(\mathrm{X} - \frac{\mathrm{D}u}{\mathrm{D}t}\right).$$

If the fluid is incompressible, we have also

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

On the same hypothesis of l being large compared with $\frac{d}{dt}$, we get for the values of the stresses,

$$\mathrm{N}_1 = p + 2\mu\frac{du}{dx},$$

$$\mathrm{T}_1 = \mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right).$$

When l is small the state of strain subsides slowly, as in Canada Balsam, in which we can see the effects of strain on polarized light. In this case the equations reduce to

$$(\kappa + \tfrac{1}{3}\nu)\frac{d\theta}{dx} - n\nabla^2 u + \rho\left(\mathrm{X} - \frac{\mathrm{D}u}{\mathrm{D}t}\right) = 0,$$

an equation of precisely the same character as the usual equation for the motion of elastic solids, except that here we have velocities instead of relative displacements.

Evidently l is the reciprocal of a time, which time, according to Prof. Clerk Maxwell, may for Canada Balsam be $\frac{1}{10}$ second. In air it is $\frac{1}{5{,}000{,}000{,}000}$ second. In iron it is perhaps about 12 hours or more for *small strains*. $\frac{1}{l}$ is called the modulus of the time of relaxation of rigidity.

The conditions to be fulfilled at the bounding surface are easy to find. For if P, Q, R be the components of the external force at any point of

the bounding surface, l, m, n be the direction cosines of the normal to the surface, we must have

$$\rho P = lN_1 + mT_3 + nT_2$$

with two other equations.

The conditions to be fulfilled when the fluid is in contact with a solid may be thus found. According to Stokes (*Camb. Trans.*, Vol. viii.), we have to express that the film of fluid in contact with the solid does not move. He reasons thus: A very small relative velocity of adjacent molecules of fluid developes a finite friction; so a finite difference of velocity between the molecules of the fluid and the fixed solid would develope an infinitely greater friction. This would make equilibrium impossible, and therefore the molecules of the fluid in contact with the solid must be at rest.

According to Helmholtz and Pietrowski, there is slipping between the fluid and the solid, but more between water and silver than between water and glass. Thus the friction on unit of surface of the solid is $\lambda \times$ (tangential velocity), where λ is constant for the same solid and the same fluid. The friction *in the fluid*, if it be moving parallel to a line on a plane surface, is

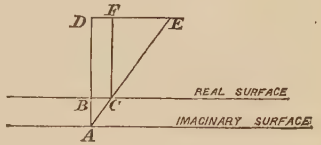
$$\mu \frac{du}{dy},$$

the line being taken as the axis of x. Hence, for equilibrium of the extreme stratum,

$$\lambda u = \mu \frac{du}{dy}.$$

Hence the phenomena are approximately the same as if an additional stratum of fluid of thickness $\frac{\mu}{\lambda}$ were added and the solid surface were such that there should be no slipping.

For if BD be a line of particles at any time, CE the same line after a time δt, and BD $= \delta y$, then



$$BC = u\delta t, \quad FE = \frac{du}{dy}\delta y \delta t,$$

and therefore

$$BA = BC \frac{1}{\frac{du}{dy}\delta t} = \frac{u}{\frac{du}{dy}} = \frac{\mu}{\lambda}.$$

7. Thomson and Tait (*Nat. Phil.*, Vol. i., Art. 741) do not appear to recognise the distinction above referred to between solids and fluids; viz., that in solids the limit of elasticity is finite, whereas in fluids it is practically zero. They speak of two kinds of resistance to change of shape. The first is molecular friction, which they explain as being an internal resistance to change of shape depending on

the rapidity of the change, and apply the term *viscosity* to resistance of this kind, whether in solids or fluids. On the views before explained, *resistance* to change of shape is an inapt term to apply to fluids; they do not resist, strictly speaking, only they are slow to move. With regard to the second kind of resistance, the same high authorities "feel bound to remark that the word viscosity, as used hitherto by the best authors, ... has been employed to designate a property of slow continual yielding under the action of continued stress;" but they prefer to speak of this property as *plasticity*. They draw no distinctions between bodies which do, and which do not, require a finite force to cause change of shape.

8. We might, however, adopt their view as to the nature of fluid friction and the resistances generated by the relative velocities of the particles, and proceed to investigate the general equations of motion of *viscous* fluids (in their sense) on the assumption that, "*When one particle is torn from another, the force called into play depends on the velocity with which they separate.*"

We may expand the force in ascending powers of the velocity, and, for slow motions at any rate, we may omit powers of the velocity above the first. Thus, if q be the relative velocity with which two particles m and m' separate, we shall have for the force exerted between them (see Haughton, Trans. of Royal Irish Academy, Vol. iii., Pt. 2),

$$f = \mathrm{F}_0 + \mathrm{F}_1 q,$$

where F_0, F_1 are functions of the original length (mm'), and implicitly of the coordinates of m, if external forces act on the system. The meaning of the term F_0 in this expression is that in fluids the molecular forces do not equilibrate themselves without the aid of external pressures.

Let (x, y, z) be coordinates of a particle m; (x', y', z') the coordinates of a neighbouring particle m'; $(x+\xi, y+\eta, z+\zeta)$ the coordinates of m when the medium is displaced in the time δt; so that, if u, v, w be the resolved velocities of m,

$$\xi = u\delta t, \quad \eta = v\delta t, \quad \zeta = w\delta t.$$

Let $\rho, \alpha, \beta, \gamma$ give the original length and direction of the line joining m and m'; ρ' the alteration in distance between m and m' after the displacement; so that

$$\rho' = q \,.\, \delta t.$$

Let V be the potential energy of all the molecular forces on any one particle, m, of the medium called into play by the separation of the neighbouring particles during the time δt; then

$$\begin{aligned} \mathrm{V} &= \Sigma \cdot \int_0^{\rho'} f \cdot d\rho'; \quad \text{but} \quad f = \mathrm{F}_0 + \mathrm{F}_1 q, \\ &\qquad\qquad\qquad\qquad d\rho' = q\,dt; \\ &= \Sigma \cdot \int_0^{\rho t} (\mathrm{F}_0 + \mathrm{F}_1 q)\, q\, dt \\ &= \delta t \iiint (\mathrm{F}_0 q + \mathrm{F}_1 q^2)\, \rho^2 \sin\theta\, d\rho\, d\theta\, d\phi, \end{aligned}$$

integrating over the sphere of molecular action. The limits clearly are

$$\begin{aligned} \phi &= 0 \quad \text{to} \quad \phi = 2\pi, \\ \theta &= 0 \quad \text{to} \quad \theta = \pi, \\ \rho &= 0 \quad \text{to} \quad \rho = \infty. \end{aligned}$$

Substitute for q its value, viz.,

$$\begin{aligned} q = \rho \Big\{ &\frac{du}{dx}\cos^2\alpha + \frac{dv}{dy}\cos^2\beta + \frac{dw}{dz}\cos^2\gamma \\ &+ U\cos\beta\cos\gamma + V\cos\gamma\cos\alpha + W\cos\alpha\cos\beta \Big\}, \end{aligned}$$

where $U = \dfrac{dv}{dz} + \dfrac{dw}{dy}$, and V, W have similar values; and, after some rather tedious reductions, we find that, if V be made symmetrical with respect to the three coordinate planes, and also with respect to the three axes (see Green, "Reflection and Refraction of Light," Memoirs, p. 251, &c.),

$$\begin{aligned} \mathrm{V} = \delta t \{ &p\,(s_1 + s_2 + s_3) + \mathrm{A}\,(s_1^2 + s_2^2 + s_3^2) + \tfrac{1}{3}\mathrm{A}\,(U^2 + V^2 + W^2) \\ &+ \tfrac{2}{3}\mathrm{A}\,(s_2 s_3 + s_3 s_1 + s_1 s_2)\}, \end{aligned}$$

where $s_1 = \dfrac{du}{dx}$, and s_2, s_3 have similar values.

We may now use Lagrange's equation of Virtual Velocities, and express that the variation of the potential energy of the internal forces called into play by the separation of the particles is equivalent to the variation of the force-function of the impressed and reversed effective forces. That is, if dm be an element of mass, ρ the density,

$$dm = \rho\, dx\, dy\, dz,$$

and

$$\begin{aligned} \iiint \Big\{ &\Big(\mathrm{X} - \frac{\mathrm{D}u}{\mathrm{D}t}\Big)\delta\xi + \Big(\mathrm{Y} - \frac{\mathrm{D}v}{\mathrm{D}t}\Big)\delta\eta + \Big(\mathrm{Z} - \frac{\mathrm{D}w}{\mathrm{D}t}\Big)\delta\zeta \Big\}\, dm \\ &= \iiint \delta\mathrm{V} \cdot dx\, dy\, dz, \end{aligned}$$

integrating over the whole fluid.

It may be observed that δt denotes the time in which m moves from position (x, y, z) to $(x+\xi, y+\eta, z+\zeta)$; while $\delta\xi$, $\delta\eta$, $\delta\zeta$, $\delta\mathrm{V}$ denote arbitrary variations.

But $\delta\xi = \delta u \,.\, \delta t$; so, eliminating δt, we have

$$\iiint \left\{ \left(\mathrm{X} - \frac{\mathrm{D}u}{\mathrm{D}t} \right) \delta u + \&c. \right\} dm$$

$$= \iiint p \left(\frac{d\,.\,\delta u}{dx} + \frac{d\,.\,\delta v}{dy} + \frac{d\,.\,\delta w}{dz} \right) dx\,dy\,dz$$

$$+ \iiint \left\{ 2\mathrm{A} \left(s_1 \frac{d\,.\,\delta u}{dx} + s_2 \frac{d\,.\,\delta v}{dy} + s_3 \frac{d\,.\,\delta w}{dz} \right) + \&c.\ \&c. \right\} dx\,dy\,dz.$$

Integrating by parts, and equating the coefficients of δu, δv, δw in the triple integrals, δ denoting *arbitrary* variations, we get, after some reductions,

$$\rho \left(\frac{\mathrm{D}u}{\mathrm{D}t} - \mathrm{X} \right) = \frac{dp}{dx} - \mu \nabla^2 u + 2\mu \frac{d\theta}{dx},$$

where $\mu = \frac{2}{3}\mathrm{A}$, and $\theta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$.

If the fluid be incompressible, and p be taken so as to denote a *push*, we have, as before,

$$\rho \left(\mathrm{X} - \frac{\mathrm{D}u}{\mathrm{D}t} \right) = \frac{dp}{dx} + \mu \nabla^2 u.$$

To express the conditions at the limits in a convenient form, we need only write

$$\mathrm{N}_1 = \frac{d\mathrm{V}}{ds_1},$$

$$\mathrm{T}_1 = \frac{d\mathrm{V}}{dU},$$

with similar expressions for N_2, N_3, T_2, T_3, and equate the coefficients of δu, δv, δw in the *double* integrals, thus getting

$$\rho \mathrm{P} = l\mathrm{N}_1 + m\mathrm{T}_3 + n\mathrm{T}_2,$$

&c.,

where P, Q, R are the components of the external forces at any point of the boundary surface, and l, m, n are the direction-cosines of the normal to the surface at this point.

The values of N_1, T_1, &c. are easy to find. They are

$$\mathrm{N}_1 = -p + 2\mu \frac{du}{dx},$$

$$\mathrm{T}_1 = \mu \left(\frac{dv}{dz} + \frac{dw}{dy} \right),$$

&c.,

Clearly, the meaning of these expressions is as follows:—They are the normal and tangential stresses called into play by the viscosity of the fluid.

9. I will now look at this function V, with which we have been dealing, from another point of view. If we suppose a body to yield to a stress, and to oppose this stress only by its innate resistance to change of shape (using the term "resistance" rather loosely), we may assert that *the work done per unit of volume by the molecular forces called into play by the strain*, or, in other words, *the function* V, *is precisely equivalent to the work done per unit of volume by distorting stress working against innate viscosity* while the substance moves from a state of no strain to the state of strain under consideration (see Thomson and Tait, Nat. Phil., I., Art. 671). That is to say, if, as in § 6, N_1, N_2, N_3; T_1, T_2, T_3 are the stresses, and s_1, s_2, s_3; U, V, W the strains,

$$V = \int \{N_1 ds_1 + N_2 ds_2 + N_3 ds_3 + T_1 dU + T_2 dV + T_3 dW\}$$
$$= \int \left\{ N_1 \frac{ds_1}{dt} + \dots \right\} dt.$$

But if the fluid be incompressible, we have, by the method and assumptions of § 6,

$$N_1 = p + 2\mu \frac{ds_1}{dt},$$
$$T_1 = \mu \frac{dU}{dt}.$$

Hence, remembering that $\frac{d(s_1+s_2+s_3)}{dt} = 0$, and writing $\frac{ds_1}{dt} = s_1'$, $\frac{dU}{dt} = U'$, &c., we get

$$V = \int \{2\mu (s_1'^2 + s_2'^2 + s_3'^2) + \mu (U'^2 + V'^2 + W'^2)\} dt,$$

which agrees with the value of V given in § 8 and founded on certain assumptions as to molecular structure.

Section II.

On the Application of the General Equations of Motion.

10. *To determine the motion of sound in a straight pipe.*

Taking the general equations of a compressible fluid given in § 6, we have, since $u = f(x, t)$, $v = 0$, $w = 0$,

$$\mu \frac{d^2u}{dx^2} - \frac{dp}{dx} = \rho \frac{Du}{Dt}$$
$$= \rho \left(\frac{du}{dt} + u \frac{du}{dx} \right)$$
$$= \rho \frac{du}{dt}, \quad \text{if } u \text{ is small,}$$
$$= D \frac{du}{dt} \quad \text{approximately,}$$

where D is the original density of the air;

ρ „ actual „

By continuity, $$\frac{1}{D}\frac{d\rho}{dt}+\frac{du}{dx}=0,$$

Also, $$p=\kappa\rho;$$

therefore, eliminating p and ρ,

$$\mu\frac{d^3u}{dx^2\,dt}+\kappa D\frac{d^2u}{dx^2}=D\frac{d^2u}{dt^2},$$

or $$\frac{d^2u}{dt^2}=\kappa\frac{d^2u}{dx^2}+\frac{\mu}{D}\frac{d^3u}{dx^2dt}.$$

To solve this, assume

$$u=f(t)\frac{\cos 2\pi x}{\lambda}+\phi(t)\frac{\sin 2\pi x}{\lambda};$$

so that $f(t)$ and $\phi(t)$ must satisfy

$$f''(t)+\frac{4\pi^2}{\lambda^2}\kappa f(t)+\frac{4\mu\pi^2}{D\lambda^2}f'(t)=0,$$

or $$f(t)=e^{-ct}\left\{C_1\cos\frac{2\pi bt}{\lambda}+C_2\sin\frac{2\pi bt}{\lambda}\right\},$$

where $$c=\frac{2\pi^2}{\lambda^2}\frac{\mu}{D},$$

$$b^2=\kappa-\frac{\pi^2\mu^2}{\lambda^2D^2};$$

so $$u=Ae^{-ct}\cos\left(\frac{2\pi}{\lambda}(bt-x)+B\right).$$

So the effect of friction is—

(1) To make the intensity of sound diminish in Geometric Progression as the time increases in Arithmetic Progression.

(2) To diminish the velocity of sound, and to make the high notes travel slowest. (*Cf.* Stokes, Camb. Trans., Vol. viii.)

11. *Discharge of water through a long straight pipe of uniform section.*

(See Boussinesq, Liouv., t. xiii., p. 383; Stokes, Camb. Trans., Vol. viii.)

We shall consider the motion to have reached a steady state. Take the axis of x in the direction of the motion, and let y be horizontal. Let the inclination of the pipe to the horizon be α, and let gravity be the only force acting.

Then $v=0$, $w=0$; therefore $\frac{du}{dx}=0$ by continuity. That is, u, and therefore also ∇^2u, is independent of x.

By substitution in the general equations in § 6, we have

$$\frac{dp}{dx} - g\rho \sin \alpha = -\mu\nabla^2 u - \rho \frac{du}{dt}$$
$$= -\mu\nabla^2 u \quad \textit{for steady motion},$$
$$\frac{dp}{dy} = 0,$$
$$\frac{dp}{dz} - g\rho \cos \alpha = 0;$$

therefore
$$\frac{d}{dy}(\mu\nabla^2 u) = 0,$$
$$\frac{d}{dz}(\mu\nabla^2 u) = 0,$$

also
$$\frac{d}{dx}(\mu\nabla^2 u) = 0;$$

therefore
$$-\mu\nabla^2 u = \frac{dp}{dx} - g\rho \sin \alpha = f(t) \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

therefore
$$p = \int \left(\frac{dp}{dx}dx + \frac{dp}{dy}dy + \frac{dp}{dz}dz\right)$$
$$= x\frac{dp}{dx} + g\rho z \cos \alpha + \phi(t) \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now, if the pressures exerted at the two extremities of the pipe be independent of the time, both $f(t)$ and $\phi(t)$ are absolute constants. Call them C and C′ respectively.

In the case of an *open pipe* or *channel*, the pressure at the free surface, if an atmospheric pressure, must be independent of x; therefore, in this case, $\frac{dp}{dx} = 0$.

In the case of a *closed pipe*, we can only assert that $\frac{dp}{dx} =$ constant $= \mathrm{A}$ suppose. Then, if p_0 be the pressure when $x=0$, p_1 the pressure when $x=l$, we have

$$\mathrm{A} = -\frac{p_0 - p_1}{l},$$

where l is the length of the pipe.

Thus the equation for u becomes

$$\frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + \mathrm{M} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where, for a *channel*, $\mathrm{M} = \frac{g\rho \sin \alpha}{\mu}$;

for a *closed pipe*, $\mathrm{M} = \frac{g\rho \sin \alpha}{\mu} + \frac{p_0 - p_1}{\mu l}$.

The boundary conditions are

$$\left.\begin{aligned} u &= 0 \quad \text{along the sides of the pipe} \\ \frac{du}{dz} &= 0 \quad \text{at the free surface (if any)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (4).$$

Let us take the special case of *a closed pipe of uniform circular section, the pipe being supposed full.*

We may assume the motion to be symmetrical round the axis, *i. e.*, the section to be so small that gravity does not render the flow unsymmetrical.

Transforming to polars, we get, since on this supposition $\frac{du}{d\theta} = 0$,

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \mathrm{M} = 0;$$

or, integrating,

$$r\frac{du}{dr} + \tfrac{1}{2}\mathrm{M}r^2 = 0,$$

no constant being added, for we could not have $\frac{du}{dr} = \infty$ when $r = 0$.

Integrating again,

$$u = \frac{\mathrm{M}}{4}(a^2 - r^2), \quad \text{since} \quad u = 0 \quad \text{when} \quad r = a.$$

$$\begin{aligned} \textit{The discharge per unit of time} &= \frac{\mathrm{M}}{4}\int_0^a \int_0^{2\pi} (a^2 - r^2)\, r\, d\theta\, dr \\ &= \frac{\mathrm{M}}{8}\pi a^4 \\ &= \frac{p_0 - p_1}{8\mu l}\pi a^4 \quad \text{if the pipe be horizontal} \end{aligned}$$

$\ldots\ldots\ldots (5).$

It is easy, as in Boussinesq, Liouv., t. xiii., to solve the problem for the cases of a pipe of elliptic or of rectangular uniform section, the only difficulty being to assume a solution of the equation in u which shall suit the case. Omitting this, I pass on to the

Experimental Verification for a Circular Pipe.

We learn from the experiments of M. Poiseuille (Comptes Rendus t. xv., p. 1167) that the discharge through a straight narrow circular pipe is proportional to the difference of pressures at the ends, the reciprocal of the length and the fourth power of the diameter of the section. This is exactly the law given by equation (5) above.

Further, the material of the tube does not affect the discharge; which again agrees with our result, which only involves μ the coefficient of internal friction of the fluid.

Boussinesq (Liouv., t. xiii.) finds, by comparing his formula with the results of experiment, that for water, taking a second as the unit of

time, a millimetre the unit of space, and the weight of a milligramme the unit of force, $\mu = \cdot 0001336$.

12. *Motion of an incompressible fluid of uniform density revolving in concentric circular annuli.*

The motion is supposed to be in two dimensions; let V be the velocity of a particle, whose polar coordinates are r, θ,

$$\begin{aligned} u &= -\mathrm{V}\sin\theta \\ v &= \mathrm{V}\cos\theta \\ w &= 0\,; \end{aligned}$$

and one equation of motion is

$$\frac{dp}{dx} + \rho\frac{\mathrm{D}u}{\mathrm{D}t} = \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}\right).$$

Supposing the motion symmetrical round the centre, then

$$\frac{d\mathrm{V}}{d\theta} = 0,$$

as indeed follows from continuity,

also $$\frac{dp}{d\theta} = 0.$$

The acceleration $= \frac{\mathrm{V}^2}{r}$ inwards along the normal, and $\frac{dp}{dx} = \frac{dp}{dr}\cos\theta$. Hence, by substitution,

$$\begin{aligned} \frac{dp}{dr}\cos\theta - \rho\frac{\mathrm{V}^2}{r}\cos\theta &= \mu\left(\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \frac{1}{r^2}\frac{d^2u}{d\theta^2}\right) \\ &= \mu\left(-\frac{d^2\mathrm{V}}{dr^2}\sin\theta - \frac{1}{r}\frac{d\mathrm{V}}{dr}\sin\theta + \frac{1}{r^2}\mathrm{V}\sin\theta\right). \end{aligned}$$

Taking the axis of x to coincide with the radius vector,

$$\frac{dp}{dr} - \rho\frac{\mathrm{V}^2}{r} = 0,$$

$$\frac{d^2\mathrm{V}}{dr^2} + \frac{1}{r}\frac{d\mathrm{V}}{dr} - \frac{\mathrm{V}}{r^2} = 0\,;$$

whence $$\mathrm{V} = \frac{\mathrm{C}}{r} + \mathrm{C}'r,$$

Let us consider the case of a fluid occupying the space between two circular cylinders having a common axis. If the internal cylinder, radius a, revolve with a velocity u, and the external cylinder, radius b, be at rest, the conditions are—

when $r = a$, $\mathrm{V} = u$, or $u = \frac{\mathrm{C}}{a} + \mathrm{C}'a$,

$r = b$, $\mathrm{V} = 0$, or $0 = \frac{\mathrm{C}}{b} + \mathrm{C}'b$;

therefore $$V = \frac{au}{b^2-a^2} \cdot \frac{b^2-r^2}{r}.$$

This might be tested by experiment by employing several different internal cylinders in turn, and making them revolve with different velocities.

This case is wrongly treated by Newton, Princip., Lib. ii., Sec. ix., Prop. li., but rightly treated by Stokes, Camb. Phil. Trans., Vol. viii.

13. *Discharge through a pipe of finite dimensions whose horizontal section is a part of two concentric circles.*

Let V be the vel. perp. to the rad. vector,

V′ the vel. in the direction of the rad. vector,

these two directions being in the horizontal plane ;

w the vel. perp. to this plane.

Then, the motion being assumed to have reached the steady state, and all the normal sections of the pipe being subjected to the same conditions, we are entitled to assume that V, V′, and w are independent of θ.

Now $$u = -V\sin\theta + V'\cos\theta,$$
$$v = V\cos\theta + V'\sin\theta,$$

and if, for shortness, we write

$$\Delta^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{d^2}{dz^2},$$

we have by transformation

$$-\nabla^2 u = -\sin\theta\left(\Delta^2 V - \frac{V}{r^2}\right) + \cos\theta\left(\Delta^2 V' - \frac{V'}{r^2}\right)$$

and $$-\nabla^2 v = \cos\theta\left(\Delta^2 V - \frac{V}{r^2}\right) + \sin\theta\left(\Delta^2 V' - \frac{V'}{r^2}\right).$$

Also

$$\frac{Du}{Dt} = u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}$$

$$= (-V\sin\theta + V'\cos\theta)\frac{du}{dx} + (V\cos\theta + V'\sin\theta)\frac{du}{dy} + w\frac{du}{dz}$$

$$= V\left(-\sin\theta\frac{d}{dx} + \cos\theta\frac{d}{dy}\right)u + V'\left(\cos\theta\frac{d}{dx} + \sin\theta\frac{d}{dy}\right)u + w\frac{du}{dz}$$

$$= V\frac{du}{rd\theta} + V'\frac{du}{dr} + w\frac{du}{dz}$$

$$= -\frac{V^2}{r}\cos\theta - \frac{VV'}{r}\sin\theta + V'\left(-\frac{dV}{dr}\sin\theta + \frac{dV'}{dr}\cos\theta\right)$$
$$+ w\left(-\frac{dV}{dz}\sin\theta + \frac{dV'}{dz}\cos\theta\right),$$

$$\frac{Dv}{Dt} = \text{similar quantity},$$

$$\frac{Dw}{Dt} = u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}$$

$$= V\left(-\sin\theta\frac{d}{dx} + \cos\theta\frac{d}{dy}\right)w + V'\left(\cos\theta\frac{d}{dx} + \sin\theta\frac{d}{dy}\right)w + w\frac{dw}{dz}$$

$$= V'\frac{dw}{dr} + w\frac{dw}{dz}, \qquad \text{since } \frac{dw}{d\theta} = 0,$$

$$\frac{dp}{dx} = \frac{dp}{dr}\cos\theta - \frac{dp}{rd\theta}\sin\theta.$$

Substituting in the general equations, and making the axis of x coincide with the direction of r, we get

$$\frac{dp}{dr} = \mu\left(\Delta^2 V' - \frac{V'}{r^2}\right) + \rho\frac{V^2}{r} - \rho V'\frac{dV'}{dr} - \rho w\frac{dV'}{dz} \quad \ldots\ldots\ldots (1),$$

$$\frac{dp}{rd\theta} = \mu\left(\Delta^2 V - \frac{V}{r^2}\right) - \rho\frac{VV'}{r} - \rho V'\frac{dV}{dr} - \rho w\frac{dV}{dz} \quad \ldots\ldots\ldots (2),$$

$$\frac{dp}{dz} = -\mu\nabla^2 w - \rho V'\frac{dw}{dr} - \rho w\frac{dw}{dz} - \rho g \quad \ldots\ldots\ldots\ldots (3).$$

Also, by continuity, $\dfrac{du}{dx} + \dfrac{dv}{dy} + \dfrac{dw}{dz} = 0,$

or, by an easy transformation,

$$\frac{dV'}{dr} + \frac{V'}{r} + \frac{dw}{dz} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Now $$\frac{d}{dr}\{r \times \text{equation (2)}\} = \frac{d}{d\theta}\{\text{equation (1)}\} = 0,$$

$$\frac{d}{dz}\{r \times \text{equation (2)}\} = \frac{d}{d\theta}\{\text{equation (3)}\} = 0,$$

Also $$\frac{d}{d\theta}\{r \times \text{equation (2)}\} = 0.$$

Hence $$\text{equation (2)} = \frac{c}{r},$$

or $$\frac{dp}{d\theta} = c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The equations, therefore, to determine the motion are (1), (3), (4), and (5). They admit of approximate solution in the case of a rectangular pipe of small altitude, which solution has been conducted by Boussinesq.

I shall take the case in which the motion is so slow that we may

neglect the inertia of the fluid, and then the equations become

$$\frac{dp}{dr} = \mu\left(\frac{d^2V'}{dr^2}+\frac{1}{r}\frac{dV'}{dr}+\frac{d^2V'}{dz^2}-\frac{V'}{r^2}\right) \quad \ldots\ldots\ldots\ldots (6),$$

$$\frac{dp}{dz} = \mu\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}+\frac{d^2w}{dz^2}\right)-\rho g \quad \ldots\ldots\ldots\ldots (7),$$

$$\frac{d(V'r)}{dr}+\frac{d(wr)}{dz} = 0 \quad \ldots\ldots\ldots\ldots (8),$$

and

$$\frac{d^2V}{dr^2}+\frac{1}{r}\frac{dV}{dr}+\frac{d^2V}{dz^2}-\frac{V}{r^2} = \frac{c}{\mu r} \quad \ldots\ldots\ldots\ldots (9).$$

I will now proceed to the solution of equations (6), (7), (8), by a method which was suggested by Stokes, Camb. Trans., Vol. ix., p. 24, *sqq.*

Eliminating p from (6) and (7), since

$$\frac{d}{dr}\Delta^2 w = \left(\Delta^2-\frac{1}{r^2}\right)\frac{dw}{dr},$$

we get

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\frac{d^2}{dz^2}-\frac{1}{r^2}\right)\left(\frac{dw}{dr}-\frac{dV'}{dz}\right) = 0 \quad \ldots\ldots\ldots (10).$$

But, by (8), we see that

$$r(w\,dr - V'\,dz) = \text{perfect differential} = d\psi \text{ say},$$

so

$$\left.\begin{aligned} w &= \frac{1}{r}\frac{d\psi}{dr} \\ V' &= -\frac{1}{r}\frac{d\psi}{dz} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (11),$$

$$\therefore \quad \frac{dw}{dr}-\frac{dV'}{dz} = \frac{1}{r}\left(\frac{d^2}{dr^2}+\frac{d^2}{dz^2}-\frac{1}{r}\frac{d}{dr}\right)\psi$$
$$= r^{-1}\chi \text{ say};$$

Thus

$$\frac{1}{r}\frac{d}{dr}\left(\frac{dw}{dr}-\frac{dV'}{dz}\right) = -r^{-3}\chi+r^{-2}\frac{d\chi}{dr},$$

and

$$\frac{d^2}{dr^2}\left(\frac{dw}{dr}-\frac{dV'}{dz}\right) = 2r^{-3}\chi-2r^{-2}\frac{d\chi}{dr}+r^{-1}\frac{d^2\chi}{dr^2};$$

$$\therefore \quad \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}+\frac{d^2}{dz^2}\right)\left(\frac{dw}{dr}-\frac{dV'}{dz}\right) = -\frac{1}{r^2}\frac{d\chi}{dr}+\frac{1}{r}\frac{d^2\chi}{dr^2}+\frac{1}{r}\frac{d^2\chi}{dz^2}$$
$$= \frac{1}{r}\left(\frac{d^2}{dr^2}+\frac{d^2}{dz^2}-\frac{1}{r}\frac{d}{dr}\right)^2\psi;$$

and equation (10) becomes

$$\left(\frac{d^2}{dr^2}+\frac{d^2}{dz^2}-\frac{1}{r}\frac{d}{dr}\right)^2\psi = 0 \quad \ldots\ldots\ldots\ldots (12).$$

Now let us put

$$r = \rho\sin\phi,$$
$$z = \rho\cos\phi,$$

then
$$\frac{d}{dr} = \frac{d}{d\rho}\sin\phi + \frac{1}{\rho}\frac{d}{d\phi}\cos\phi,$$
$$\frac{d}{dz} = \frac{d}{d\rho}\cos\phi - \frac{1}{\rho}\frac{d}{d\phi}\sin\phi,$$

and $\therefore$
$$\frac{d^2}{dr^2} + \frac{d^2}{dz^2} - \frac{1}{r}\frac{d}{dr} = \frac{d^2}{d\rho^2} + \frac{1}{\rho^2}\frac{d^2}{d\phi^2} - \frac{1}{\rho^2}\frac{d}{d\phi}\cot\phi$$
$$= \frac{d^2}{d\rho^2} + \frac{1}{\rho^2}\sin\phi\,\frac{d}{d\phi}\left(\frac{1}{\sin\phi}\frac{d}{d\phi}\right).$$

Now the equation for Laplace's functions, symmetrical round an axis, is
$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\,\frac{dP_i}{d\phi}\right) + i(i+1)P_i = 0,$$

so, if D′ denote the operator $\sin\phi\,\frac{d}{d\phi}\left(\frac{1}{\sin\phi}\frac{d}{d\phi}\right)$, we see that
$$D'\left(\sin\phi\,\frac{dP_i}{d\phi}\right) = -i(i+1)\sin\phi\,\frac{dP_i}{d\phi};$$

or, if $\sin\phi\,\frac{dP_i}{d\phi} = Q_i$,
$$\{D' + i(i+1)\}Q_i = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (13).$$

The symbolical form of (12) being then
$$\left(\frac{d^2}{d\rho^2} + \frac{D'}{\rho^2}\right)^2\psi = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (14),$$

we are led to assume $\psi = Q_i f(\rho)$.

Substituting in equation (14), we find, since $\frac{d^2}{d\rho^2}$ does not operate on Q_i, and D′ does not operate on $f(\rho)$,
$$\left(\frac{d^2}{d\rho^2} + \frac{D'}{\rho^2}\right)\psi = Q_i\left(\frac{d^2}{d\rho^2} - \frac{i(i+1)}{\rho^2}\right)f(\rho),$$

so
$$\left(\frac{d^2}{d\rho^2} + \frac{D'}{\rho^2}\right)^2\psi = Q_i\left(\frac{d^2}{d\rho^2} - \frac{i(i+1)}{\rho^2}\right)^2 f(\rho),$$

and finally
$$\left(\frac{d^2}{d\rho^2} - \frac{i(i+1)}{\rho^2}\right)^2 f(\rho) = 0.$$

To solve this, put
$$\left(\frac{d^2}{d\rho^2} - \frac{i(i+1)}{\rho^2}\right)f(\rho) = F(\rho),$$

then
$$\left(\frac{d^2}{d\rho^2} - \frac{i(i+1)}{\rho^2}\right)F(\rho) = 0,$$

or
$$F(\rho) = A\rho^{i+1} + B\rho^{-i};$$

$\therefore$
$$\left\{\rho^2\frac{d^2}{d\rho^2} - i(i+1)\right\}f(\rho) = A\rho^{i+3} + B\rho^{-(i-2)},$$

$$\therefore \qquad f(\rho) = C\rho^{i+1} + D\rho^{-i} + A'\rho^{i+3} + B'\rho^{-(i-2)}.$$

Hence $$\psi = \sin\phi \frac{dP_i}{d\phi} (C\rho^{i+1} + D\rho^{-i} + A'\rho^{i+3} + B'\rho^{-(i+2)}) \ldots\ldots\ldots (15).$$

To find the conditions necessary to be fulfilled at the boundaries.

Let a and b be the internal and external radii of the section of the pipe made by the horizontal plane of xy. The fluid in contact with the sides of this section must be at rest.

Thus, when $\left.\begin{matrix} z = 0 \\ r = a \\ \text{or } b \end{matrix}\right\}$ we must have $\left.\begin{matrix} V' = 0 \\ w = 0 \end{matrix}\right\}$,

which, by (11), requires that, for these values of r and z,

$$\left.\begin{matrix} \frac{d\psi}{dr} = 0 \\ \frac{d\psi}{dz} = 0 \end{matrix}\right\};$$

that is, when $\left.\begin{matrix} \phi = \frac{\pi}{2} \\ \rho = a \\ \text{or } b \end{matrix}\right\}$ we must have $\left.\begin{matrix} \frac{d\psi}{d\rho} = 0 \\ \frac{d\psi}{d\phi} = 0 \end{matrix}\right\}$.

The condition $\frac{d\psi}{d\rho} = 0$ gives

$$\left.\begin{matrix} \left[\sin\phi \frac{dP_i}{d\phi}\right]_{\phi=\frac{\pi}{2}} f'(a) = 0 \\ \text{and} \qquad \left[\sin\phi \frac{dP_i}{d\phi}\right]_{\phi=\frac{\pi}{2}} f'(b) = 0 \end{matrix}\right\}.$$

If i is even, the expression vanishes of itself.

If i is odd, we must have $\left.\begin{matrix} f'(a) = 0 \\ \text{and} \qquad f'(b) = 0 \end{matrix}\right\}$ .....(16).

The condition $\frac{d\psi}{d\phi} = 0$ gives

$$\left.\begin{matrix} -i(i+1)\left[\sin\phi\, P_i\right]_{\phi=\frac{\pi}{2}} f(a) = 0 \\ \text{and} \qquad -i(i+1)\left[\sin\phi\, P_i\right]_{\phi=\frac{\pi}{2}} f(b) = 0 \end{matrix}\right\}.$$

If i is even, we require $\left.\begin{matrix} f(a) = 0 \\ f(b) = 0 \end{matrix}\right\}$ ..... (17).

If i is odd, the expression vanishes of itself.

Suppose, as a particular case, that $i = 1$; then

$$f(\rho) = C\rho^2 + D\rho^{-1} + A'\rho^4 + B'\rho,$$

and therefore (16) gives

$$2Ca-\frac{D}{a^2}+4A'a^3+B'=0,$$

$$2Cb-\frac{D}{b^2}+4A'b^3+B'=0;$$

$$\left.\begin{aligned}-2C&=(a+b)\frac{D}{a^2b^2}+4A'(a^2+ab+b^2)\\ B'&=(a^2+ab+b^2)\frac{D}{a^2b^2}+4A'ab(a+b)\end{aligned}\right\}\quad\text{.....(18),}$$

and

$$\psi=\sin^2\phi\left(C\rho^2+\frac{D}{\rho}+A'\rho^4+B'\rho\right)\quad\text{..... (19),}$$

where there are only two constants A′ and D left undetermined.

If we wish to determine these constants, we may take the case of a pipe uniform and circular in section everywhere, the radius of the circular section being h.

We have then the further conditions that

$$\left.\begin{aligned}r&=\frac{a+b}{2}\\ z&=\pm h\end{aligned}\right\}\quad\text{when}\quad\left.\begin{aligned}V'&=0\\ w&=0\end{aligned}\right\}\quad\text{then and};$$

$$\left.\begin{aligned}\rho&=\sqrt{\left(\frac{a+b}{2}\right)^2+h^2}=m\ \text{ say}\\ \tan\phi&=\pm\frac{a+b}{h}\end{aligned}\right\}\quad\text{i.e., when}\quad\left.\begin{aligned}\frac{d\psi}{d\rho}&=0\\ \frac{d\psi}{d\phi}&=0\end{aligned}\right\}\quad\text{then}.$$

This gives

$$\left.\begin{aligned}2Cm-\frac{D}{m^2}+4A'm^3+B'&=0\\ Cm^2+\frac{D}{m}+A'm^4+B'm&=0\end{aligned}\right\}\quad\text{.....(20).}$$

Equations (18), (19), (20) completely determine the value of ψ, and then, by (11), we deduce the values of V′ and w.

Section III.

On the Motion of Plastico-Viscous Fluids.

14. I will now return to the more general problem of *plastico-viscous* fluids referred to above in § 5. I will take the case of a body which contains A groups of a certain class, and also B groups. The forces developed due to the B groups are of the same nature as has been found before; that is to say, if we suppose the viscosity of these groups to be small, we find, as in § 6, that the forces developed depend partly on the relative velocities of the parts of the medium, partly on the state of contraction or dilatation. The forces developed due to the A groups depend on strains, which, when they increase up to a certain value,

cause rupture of the material. It is at this point that M. Tresca ("Mémoires des Savants étrangers," Tom. xx.) takes up the problem. He appears to consider that when a body of the plastic type is subject to a breaking strain, a new kind of resistance to motion is called into play—viz., a constant resistance, in no wise dependent on the relative motion of the contiguous portions of the medium. In comparison with these constant resistances, he neglects the forces (above referred to) which depend on the strains of the medium, and which may be called *elastic* forces. In his own words, "The forces which continue to produce slidings, dilatations, &c. are independent of the slidings and dilatations which have already occurred, the elasticity having so broken down that it only produces forces which may be neglected in comparison with the others." M. Tresca notices the analogy of this constant resistance to the ordinary friction between solid bodies. He proposes to call the forces developed by the motion *fluid pressures*. These fluid pressures are not, however, propagated throughout the interior of the body without alteration (as is usually supposed in the case of liquids); they suffer degradation in the transmission, and there is, in fact, a certain "sphere of activity" beyond which they cannot be transmitted at all.

15. We may now form the equations of motion of such a body. Leaving out for the present the forces due to the viscosity of the medium (the B groups), we have two facts to express:—

1°. When the tangential component of the pressure is greatest, it has for intensity the constant value K.

2°. When the tangential stress is a maximum, the relative velocity of sliding is also a maximum.

Suppose p_1, p_2, p_3 to be the three principal pressures at any point of the medium; and take these directions as axes. Then, if we look for the tangential action T, in any direction whose direction-cosines are m, m', m'', across a plane whose direction-cosines are n, n', n''; we have, since the three components of the stresses on the plane (n, n', n'') are $np_1, n'p_2, n''p_3$,* $\quad T = mnp_1 + m'n'p_2 + m''n''p_3$.

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* I am indebted to Mr. G. H. Darwin, of Trinity College, Cambridge, for the following proof:—

Take the stress quadric, and let r be the radius vector of the surface perpendicular to the plane across which we require the tangential stress.

Let p be the perpendicular on the tangent plane at the extremity of r. The resultant stress across the plane $= \frac{1}{pr}$, and acts along the perpendicular p.

The resultant tangential stress $T = \frac{1}{pr}\sqrt{1-\frac{p^2}{r^2}}$;

and if l, m, n are the direction-cosines of r, we shall find that

$$T^2 = l^2m^2(N_1-N_2)^2 + m^2n^2(N_2-N_3)^2 + n^2l^2(N_3-N_1)^2.$$

We desire to find the values of $mm'm''$, $nn'n''$, which make this a maximum, subject to the conditions

$$m^2+m'^2+m''^2=1,$$
$$n^2+n'^2+n''^2=1.$$

Differentiate partially; then, by the usual rules, we must have, if λ be an indeterminate multiplier,

$$\frac{d\mathrm{T}}{dm}+\lambda m=0,$$
$$\frac{d\mathrm{T}}{dm'}+\lambda m'=0,$$
$$\frac{d\mathrm{T}}{dm''}+\lambda m''=0;$$

and hence $$m\frac{d\mathrm{T}}{dm}+m'\frac{d\mathrm{T}}{dm'}+m''\frac{d\mathrm{T}}{dm''}+\lambda=0,$$

or $$\mathrm{T}+\lambda=0;$$

whence $$\frac{1}{m}\frac{d\mathrm{T}}{dm}=\frac{1}{m'}\frac{d\mathrm{T}}{dm'}=\frac{1}{m''}\frac{d\mathrm{T}}{dm''}=\mathrm{T},$$

or $$\frac{n}{m}p_1=\frac{n'}{m'}p_2=\frac{n''}{m''}p_3=\mathrm{T}.$$

Similarly, by differentiating with respect to n, n', n'',

$$\frac{m}{n}p_1=\frac{m'}{n'}p_2=\frac{m''}{n''}p_3=\mathrm{T},$$

which requires that $$n^2=m^2,$$
$$n'^2=m'^2,$$
$$n''^2=m''^2,$$

or $$n=\pm m,\quad n'=\pm m',\quad n''=\pm m''.$$

But $$mn+m'n'+m''n''=0;$$

$\therefore$ $$\pm m^2\pm m'^2\pm m''^2=0;$$

$\therefore$ either (1), $m=0$, $m'^2=m''^2=\frac{1}{2}$, $\mathrm{T}=\frac{1}{2}(p_2\sim p_3)$
or (2), $m'=0$, $m^2=m''^2=\frac{1}{2}$, $\mathrm{T}=\frac{1}{2}(p_3\sim p_1)$
or (3), $m''=0$, $m^2=m'^2=\frac{1}{2}$, $\mathrm{T}=\frac{1}{2}(p_1\sim p_2)$ } .....(1);

Make this a maximum, subject to $l^2+m^2+n^2=1$;

$\therefore$ $$m^2(\mathrm{N}_1-\mathrm{N}_2)^2+n^2(\mathrm{N}_3-\mathrm{N}_1)^2=l^2(\mathrm{N}_1-\mathrm{N}_2)^2+n^2(\mathrm{N}_2-\mathrm{N}_3)^2$$
$$=m^2(\mathrm{N}_1-\mathrm{N}_2)^2+n^2(\mathrm{N}_2-\mathrm{N}_3)^2.$$

These are satisfied by

$$\left.\begin{array}{lll} l^2=\frac{1}{2}, & m^2=\frac{1}{2}, & n^2=0 \\ l^2=\frac{1}{2}, & m^2=0, & n^2=\frac{1}{2} \\ l^2=0, & m^2=\frac{1}{2}, & n^2=\frac{1}{2} \end{array}\right\},$$

as in the text.

that is, the direction of the greatest tangential pressure bisects the angle between some two of the principal pressures, and the magnitude of this tangential action is equal to half the difference between those two principal pressures. Hence, if we form the cubic whose roots are the squares of the differences of the principal pressures, this equation must be satisfied by $4K^2$ being substituted for the unknown, K being the constant resistance as before.

Now let us take any axes, and let N_1, N_2, N_3; T_1, T_2, T_3 be the normal and tangential pressures as usual. Then the equation which gives the three principal pressures is, as in the ordinary Theory of Elasticity,

$$x^3 - x^2(N_1+N_2+N_3) + x(N_2N_3+N_3N_1+N_1N_2 - T_1^2 - T_2^2 - T_3^2) - (N_1T_1^2 + N_2T_2^2 + N_3T_3^2 - N_1N_2N_3 - 2T_1T_2T_3) = 0;$$

or, by getting rid of the second term,

$$x^3 + qx - r = 0,$$

where

$$\left.\begin{aligned} q &= N_2'N_3' + N_3'N_1' + N_1'N_2' - T_1^2 - T_2^2 - T_3^2 \\ r &= N_1'T_1^2 + N_2'T_2^2 + N_3'T_3^2 - N_1'N_2'N_3' - 2T_1T_2T_3 \\ N_1' &= N_1 - \tfrac{1}{3}(N_1+N_2+N_3) \\ N_2',\ N_3' &= \text{similar quantities} \end{aligned}\right\} \quad \ldots\ldots\ldots(2).$$

To form the equation whose roots are the squares of the differences of the roots of this equation, we put

$$y = (\alpha-\beta)^2 = -2q - \frac{2r}{x} - x^2,$$

or

$$x^3 + (2q+y)x + 2r = 0.$$

Also,

$$x^3 + qx - r = 0;$$

$\therefore$

$$(q+y)x + 3r = 0, \quad \text{or} \quad x = \frac{-3r}{q+y}.$$

Substituting, and putting $4K^2$ for y, we get

$$4(4K^2+q)^2(K^2+q) + 27r^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots(3).$$

This gives one equation.

16. We now desire to express the fact that, where the tangential pressure is a maximum, the relative velocity of sliding is also a maximum.

The tangential pressure across a plane (n, n', n'') in the direction (m, m', m'') is given by

$$T = mnN_1 + m'n'N_2 + m''n''N_3 + (m'n'' + m''n')T_1 + (m''n + mn'')T_2 + (m'n + mn')T_3.$$

To find the velocity of sliding in the same direction across the same plane, we may take the direction m, m', m'' as the new axis of x, and

n, n', n'' as the new axis of y; then, by the annexed diagram, it is easy to see that if u', v', w' be the velocities parallel to the new axes,

$$\begin{aligned} u' &= mu + m'v + m''w, \\ v' &= nu + n'v + n''w, \\ w' &= pu + p'v + p''w, \end{aligned}$$

	x	y	z
x'	m	m'	m''
y'	n	n'	n''
z'	p	p'	p''

with similar expressions for x', y', z' in terms of x, y, z.

Hence relative velocity of sliding of two strata parallel to (y', z'), resolved in the direction of x' $\Big\} = \dfrac{dv'}{dx'} + \dfrac{du'}{dy'}$

$$= \left(\frac{dv'}{dx}\frac{dx}{dx'} + \frac{dv'}{dy}\frac{dy}{dx'} + \frac{dv'}{dz}\frac{dz}{dx'}\right) + \left(\frac{du'}{dx}\frac{dx}{dy'} + \frac{du'}{dy}\frac{dy}{dy'} + \frac{du'}{dz}\frac{dz}{dy'}\right),$$

and

$$\begin{aligned} x &= mx' + ny' + pz', \\ y &= m'x' + \&c., \\ z &= m''x' + \&c.; \end{aligned}$$

$\therefore$ relative velocity of sliding

$$= \left(n\frac{du}{dx} + n'\frac{dv}{dx} + n''\frac{dw}{dx}\right)m + \left(n\frac{du}{dy} + n'\frac{dv}{dy} + n''\frac{dw}{dy}\right)m'$$
$$+ \left(n\frac{du}{dz} + n'\frac{dv}{dz} + n''\frac{dw}{dz}\right)m'' + 3 \text{ similar terms}$$

$$= 2mn\frac{du}{dx} + 2m'n'\frac{dv}{dy} + 2m''n''\frac{dw}{dz}$$
$$+ (m'n'' + m''n')\left(\frac{dv}{dz} + \frac{dw}{dy}\right) + (n''n + mn'')\left(\frac{dw}{dx} + \frac{du}{dz}\right)$$
$$+ (n'n + mn')\left(\frac{du}{dy} + \frac{dv}{dx}\right),$$

an expression of precisely similar form to the expression for T, but with $2\dfrac{du}{dx}$, &c. written instead of N_1, &c.

We have already shown that the direction of greatest tangential pressure bisects the angle* between some two of the principal pressures, and it is clear mechanically that these two principal pressures must be the greatest and least principal pressures; that is, the greatest tangential pressure bisects the angle between the greatest and least axes of the ellipsoid

$$N_1x^2 + N_2y^2 + N_3z^2 + 2T_1yz + 2T_2zx + 2T_3xy = C.$$

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* In this part of the proof I have received valuable assistance from Mr. G. H. Darwin.

Similarly, it follows that the direction of greatest relative velocity of sliding bisects the angle between the greatest and least axes of the ellipsoid

$$2\frac{du}{dx}x^2+2\frac{dv}{dy}y^2+2\frac{dw}{dz}z^2+2\left(\frac{dv}{dz}+\frac{dw}{dy}\right)yz+2\left(\frac{dw}{dx}+\frac{du}{dz}\right)zx+2\left(\frac{du}{dy}+\frac{dv}{dx}\right)xy=\mathrm{C}'.$$

By hypothesis, the direction in which the tangential pressure is greatest is also that in which the relative velocity of sliding is greatest; hence the axes of these two last-written ellipsoids must be coincident. The axes of the former are given by

$$\left.\begin{aligned}(\mathrm{N}_1-\lambda)l+m\mathrm{T}_3+n\mathrm{T}_2&=0\\ l\mathrm{T}_3+(\mathrm{N}_2-\lambda)m+n\mathrm{T}_1&=0\\ l\mathrm{T}_1+m\mathrm{T}_2+n(\mathrm{N}_3-\lambda)&=0\end{aligned}\right\}.$$

The axes of the latter are given by

$$\left.\begin{aligned}\left(2\frac{du}{dx}-\lambda'\right)l+\left(\frac{du}{dy}+\frac{dv}{dx}\right)m+\left(\frac{dw}{dx}+\frac{du}{dz}\right)n&=0\\ \&\text{c.}\qquad\qquad\qquad\qquad\&\text{c.}&\end{aligned}\right\}.$$

Expressing that these give identical values of l, m, n, we have

$$\left.\begin{aligned}\frac{\mathrm{N}_1-\lambda}{2\frac{du}{dx}-\lambda'}=\frac{\mathrm{T}_3}{\frac{du}{dy}+\frac{dv}{dx}}=\frac{\mathrm{T}_2}{\frac{dw}{dx}+\frac{du}{dz}}\\ \&\text{c.}\qquad\qquad\&\text{c.}\end{aligned}\right\},$$

or

$$\frac{\mathrm{N}_1-\mathrm{N}_3}{2\left(\frac{du}{dx}-\frac{dw}{dz}\right)}=\frac{\mathrm{N}_2-\mathrm{N}_3}{2\left(\frac{dv}{dy}-\frac{dw}{dz}\right)}=\frac{\mathrm{T}_1}{\frac{dv}{dz}+\frac{dw}{dy}}=\frac{\mathrm{T}_2}{\frac{dw}{dx}+\frac{du}{dz}}=\frac{\mathrm{T}_3}{\frac{du}{dy}+\frac{dv}{dx}}\qquad\ldots\ldots\ldots(4).$$

The ordinary equations deduced as in the Theory of Elasticity are

$$\frac{d\mathrm{N}_1}{dx}+\frac{d\mathrm{T}_3}{dy}+\frac{d\mathrm{T}_2}{dz}=\rho\left(\frac{\mathrm{D}u}{\mathrm{D}t}-\mathrm{X}\right)\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

$$\frac{d\mathrm{T}_3}{dx}+\frac{d\mathrm{N}_2}{dy}+\frac{d\mathrm{T}_1}{dz}=\rho\left(\frac{\mathrm{D}v}{\mathrm{D}t}-\mathrm{Y}\right)\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

$$\frac{d\mathrm{T}_2}{dx}+\frac{d\mathrm{T}_1}{dy}+\frac{d\mathrm{N}_3}{dz}=\rho\left(\frac{\mathrm{D}w}{\mathrm{D}t}-\mathrm{Z}\right)\ldots\ldots\ldots\ldots\ldots\ldots(7);$$

and, by continuity,

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

supposing the fluid to be incompressible;

and, by equation (3),

$$4(4\mathrm{K}^2+q)^2(\mathrm{K}^2+q)+27r^2=0\ldots\ldots\ldots\ldots(9).$$

These five equations, together with the four equations numbered (4), are sufficient to determine N_1, N_2, N_3, T_1, T_2, T_3, u, v, w.

These equations were given without proof in Liouv., T. xvi., p. 370.

We can, if we choose, add to equations (5), (6), (7) parts of the forces developed by the action of viscosity, as found in § 6; that is, we must add to the left-hand sides the terms $\frac{-dp}{dx}-\mu\nabla^2 u$, $\frac{-dp}{dx}-\mu\nabla^2 v$, $\frac{-dp}{dz}-\mu\nabla^2 w$ respectively.

Case of two Dimensions.

We put $v=0$, $\frac{d}{dy}=0$; $\therefore\ T_1=T_3=0$;

and, by (4),
$$\frac{N_1-N_3}{\frac{du}{dx}-\frac{dw}{dz}}=\frac{N_2-N_3}{-\frac{dw}{dz}};$$

but, by (8),
$$\frac{du}{dx}+\frac{dw}{dz}=0,$$

$\therefore$
$$N_2=\frac{N_1+N_3}{2}.$$

Hence, looking at (2),
$$N_1'=N_1-\tfrac{1}{2}(N_1+N_3)=\frac{N_1-N_3}{2},$$
$$N_2'=0,$$
$$N_3'=-\frac{N_1-N_3}{2};$$

$\therefore$
$$q=-T_2^2-N_1'^2=-T_2^2-\tfrac{1}{4}(N_1-N_3)^2,$$
$$r=0.$$

Hence (9) becomes $4T_2^2+(N_1-N_3)^2=4K^2$ .....(10),

(4) becomes
$$\frac{N_1-N_3}{\frac{du}{dx}-\frac{dw}{dz}}=\frac{2T_2}{\frac{du}{dz}+\frac{dw}{dx}} \quad \text{..... (11);}$$

and by taking in the additional terms (5) and (7) become
$$\frac{-dp}{dx}+\frac{dN_1}{dx}+\frac{dT_2}{dz}-\mu\nabla^2 u=\rho\left(\frac{Du}{Dt}-X\right) \quad \text{.....(12),}$$
$$\frac{-dp}{dz}+\frac{dT_2}{dx}+\frac{dN_3}{dz}-\mu\nabla^2 w=\rho\left(\frac{Dw}{Dt}-Z\right) \quad \text{.....(13);}$$

also,
$$\frac{du}{dx}+\frac{dw}{dz}=0 \quad \text{..... (14).}$$

17. *Discharge of a plastic body through a straight pipe of constant section inclined at a small angle* (α) *to the horizon.*

Take, as in § 11, the axis of x in the direction of the motion, y horizontal. Then $v = 0$, $w = 0$, $\frac{du}{dx} = 0$;

$$\therefore \quad T_1 = 0, \quad \text{and} \quad N_1 = N_2 = N_3 = p, \text{ say};$$

also, $$N_1' = N_2' = N_3' = 0, \quad q = -(T_2^2 + T_3^2), \quad r = 0.$$

Equation (9) of § 16 becomes

$$T_2^2 + T_3^2 = 4K^2,$$

and, by (4),

$$\frac{T_2}{\frac{du}{dz}} = \frac{T_3}{\frac{du}{dy}};$$

$$\therefore \quad T_2 = 2K\lambda \frac{du}{dz},$$

$$T_3 = 2K\lambda \frac{du}{dy},$$

where

$$\lambda = \frac{1}{\sqrt{\left(\frac{du}{dz}\right)^2 + \left(\frac{du}{dy}\right)^2}}.$$

Thus, since u is independent of x, T_2 and T_3 are so likewise. Hence, by substitution in (5), (6), and (7), we get, omitting the forces due to the inertia of fluid,

$$\frac{dp}{dx} - g\rho \sin\alpha = \frac{d}{dy}\left(2K\lambda \frac{du}{dy}\right) + \frac{d}{dz}\left(2K\lambda \frac{du}{dz}\right) \quad \ldots\ldots\ldots (1),$$

$$\frac{dp}{dy} = 0 \quad \ldots\ldots\ldots (2),$$

$$\frac{dp}{dz} - g\rho \cos\alpha = 0 \quad \ldots\ldots\ldots (3).$$

The equations of § 11 were

$$\frac{dp}{dx} - g\rho \sin\alpha = \frac{d}{dy}\left(\mu \frac{du}{dy}\right) + \frac{d}{dz}\left(\mu \frac{du}{dz}\right),$$

$$\frac{dp}{dy} = 0,$$

$$\frac{dp}{dz} - g\rho \cos\alpha = 0,$$

so that the difference consists in the presence here of the variable factor $2K\lambda$ instead of the constant factor μ.

Since T_2, T_3 are independent of x,

$$\frac{d}{dx}[\text{equation (1)}] = 0;$$

by (2), $$\frac{d}{dy}[\text{equation (1)}] = 0;$$

by (3), $$\frac{d}{dz}[\text{equation (1)}] = 0;$$

$$\therefore \quad \frac{dp}{dx} = c, \text{ where } c \text{ is a constant.}$$

So, by (1), $$\lambda\left(\frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) + \frac{d\lambda}{dy}\frac{du}{dy} + \frac{d\lambda}{dz}\frac{du}{dz} = \frac{c - g\rho\sin\alpha}{2K} \quad \ldots\ldots\ldots (4).$$

The value of c is easily found to be the difference of pressures at the two extremities of the pipe divided by the length.

Take the particular case of a closed circular pipe full of a plastico-viscous substance.

Here we must insert a term $-\mu\nabla^2 u$ on the left-hand side of equation (4), but may still neglect the inertia of the substance.

Let us, as in § 11, suppose the motion to be symmetrical round the axis; then, transforming to polars by putting $y = r\cos\theta$, $z = r\sin\theta$, we have, since we suppose $\frac{du}{d\theta} = 0$,

$$\frac{du}{dy} = \frac{du}{dr}\cos\theta,$$

$$\frac{du}{dz} = \frac{du}{dr}\sin\theta,$$

$$\lambda = \frac{1}{\frac{du}{dr}};$$

$$\therefore \quad \frac{d\lambda}{dy} = \frac{d\lambda}{dr}\cos\theta = -\frac{\frac{d^2u}{dr^2}}{\left(\frac{du}{dr}\right)^2}\cos\theta,$$

$$\frac{d\lambda}{dz} = -\frac{\frac{d^2u}{dr^2}}{\left(\frac{du}{dr}\right)^2}\sin\theta.$$

Hence $$\frac{d\lambda}{dy}\frac{du}{dy} + \frac{d\lambda}{dz}\frac{du}{dz} = -\frac{\frac{d^2u}{dr^2}}{\frac{du}{dr}};$$

and equation (4) becomes, inserting the term due to viscosity,

$$\frac{1}{\frac{du}{dr}}\left(\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr}\right) - \frac{\frac{d^2u}{dr^2}}{\frac{du}{dr}} + \mu\left(\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr}\right) = \frac{c - g\rho\sin\alpha}{2K},$$

or $$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\frac{1}{\mu r}=\frac{c-g\rho\sin\alpha}{2K\mu};$$

or, integrating twice, and remembering that $u=0$ when $r=a$,

$$u=-\frac{c-g\rho\sin\alpha}{4\mu K}(a^2-r^2)+\frac{a-r}{\mu}.$$

18. *Motion of a plastico-viscous substance revolving in concentric circular annuli.*

We take the motion in two dimensions; then, if (x, y) are the coordinates of any point in the plane we are considering, equations (10), (11), (12), (13), (14) of § 16 give, neglecting inertia,

$$\frac{dN_x}{dx}+\frac{dT}{dz}-\mu\nabla^2u=0,$$
$$\frac{dT}{dx}+\frac{dN_y}{dy}-\mu\nabla^2v=0,$$

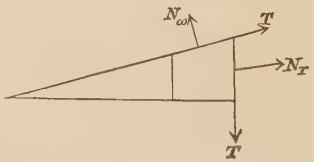
where we may include the p in N_x and N_y;

$$4T^2+(N_y-N_x)^2=4K^2,$$
$$\frac{N_y-N_x}{\frac{dv}{dy}-\frac{du}{dx}}=\frac{2T}{\frac{dv}{dx}+\frac{du}{dy}},$$
$$\frac{du}{dx}+\frac{dv}{dy}=0.$$

We desire to transform to polars, having given $u=-V\sin\theta$, $v=V\cos\theta$ (*cf.* § 12).

We cannot transform the differentials of the stresses into polar stresses by ordinary transformation of coordinates, as N_r, N_w, T, the new stresses, are stresses over a differently shaped elementary solid from the (N_x, N_y) solid.

Take, then, an element shaped as indicated, and let N_r, T be the normal and tangential stresses across an element $r\delta\theta$; N_w, T those across δr.



Resolving along and perpendicular to r, we have

$$\frac{d}{dr}(N_r r\delta\theta)\,\delta r-N_w\delta\theta\,\delta r+\frac{d}{d\theta}(T\delta r)\,\delta\theta,$$

and $$\frac{d}{d\theta}(N_w\delta r)\,\delta\theta+\frac{d}{dr}(Tr\delta\theta)\,\delta r,$$

or $$r\,\delta\theta\,\delta r\left(\frac{dN_r}{dr}+\frac{N_r-N_w}{r}+\frac{1}{r}\frac{dT}{d\theta}\right),$$

and $$r\delta\theta\,\delta r\left(\frac{dN_w}{r\,d\theta}+\frac{dT}{dr}+\frac{T}{r}\right).$$

Also, $$-\nabla^2 u = -\sin\theta\left(\frac{d^2V}{dr^2}+\frac{1}{r}\frac{dV}{dr}-\frac{V}{r^2}\right),$$

$$-\nabla^2 v = \cos\theta\left(\frac{d^2V}{dr^2}+\frac{1}{r}\frac{dV}{dr}-\frac{V}{r^2}\right),$$

$$\frac{dv}{dx}+\frac{du}{dy}=\left(\frac{dV}{dr}+\frac{V}{r}\right)\cos 2\theta,$$

$$\frac{dv}{dy}-\frac{du}{dx}=\left(\frac{dV}{dr}-\frac{V}{r}\right)\sin 2\theta.$$

Taking the axis of x, which is perfectly arbitrary, in the direction of r, and substituting in the original equations,

$$\frac{dN_r}{dr}+\frac{1}{r}\frac{dT}{d\theta}+\frac{N_r-N_w}{r}=0,$$

$$\frac{dT}{dr}+\frac{1}{r}\frac{dN_w}{d\theta}+\frac{T}{r}+\mu\left(\frac{d^2V}{dr^2}+\frac{1}{r}\frac{dV}{dr}-\frac{V}{r^2}\right)=0,$$

$$\frac{N_w-N_r}{0}=\frac{2T}{\frac{dV}{dr}+\frac{T}{r}},$$

whence $N_w = N_r$, and $T = K$. So, putting $N_r = -p$, we have finally

$$\frac{dp}{dr}=0,$$

$$\frac{dp}{r\,d\theta}=\frac{K}{r}+\mu\left(\frac{d^2V}{dr^2}+\frac{1}{r}\frac{dV}{dr}-\frac{V}{r^2}\right).$$

By differentiation, and eliminating p, we get

$$r^2\frac{d^2V}{dr^2}+r\frac{dV}{dr}-V=\left(c-\frac{K}{\mu}\right)r=c'r \text{ say.}$$

Integrating, and supposing that the space considered is the space between two infinite cylinders, radii a and b, so that $V=0$ when $r=a$ and when $r=b$, we get

$$V=\frac{c'r}{2(b^2-a^2)}\left\{b^2\log\frac{r}{b}-a^2\log\frac{r}{a}+\frac{a^2b^2}{r^2}\log\frac{b}{a}\right\}.$$

An easy Method of finding the Invariant Equation expressing any poristic relation between two Conics. By Prof. J. Wolstenholme.

[*Read December* 14*th*, 1876.]

The invariant equation expressing a poristic relation between two conics can be generally readily found by considering the limiting form when the poristic triangle or polygon passes through one or two of the common points of the two conics. The application of this method is exemplified in the following article.

1. Triangles inscribed in S and self-conjugate to S′.—In this case, if P be any point on S, and the polar of P with respect to S′ meet S in Q, R, the triangle PQR will be self-conjugate to S′; hence, when P approaches A, a common point of S and S′, one of the two points Q, R also approaches A, and if AB be a chord of S touching S′ at A, the triangle AAB inscribed in S will be self-conjugate to S′, or if AC be the chord of S′ touching S at A, BC will touch S′ at C. Hence the equations referred to the triangle ABC may be taken

$$S \equiv z^2+2fyz+2hxy = 0, \quad S' \equiv y^2-2zx = 0,$$

and the discriminant of $kS+S'$ is $k^3h^2+2k^2fh+1$, so that $\Theta' = 0$.

2. In exactly the same way, interchanging points and lines, if a common tangent touch S in B and S′ in C, and CA be a chord of S′ touching S, then, if BA touch S′ in A, the triangle formed by BC taken twice and CA will be a triangle self-conjugate to S′ and circumscribing S. The equations will then be

$$S \equiv x^2+n^2z^2+2nzx+2hxy = 0, \quad S' \equiv y^2+2zx = 0,$$

and the discriminant will be

$$k^2n^2-(kn+1)^2-k^3n^2h^2, \text{ or } k^3n^2h^2+2kn+1,$$

so that $\Theta = 0$.

Hence it appears that $\Theta = 0$ is the condition that triangles can be circumscribed to S which are self-conjugate to S′, and at the same time that triangles can be inscribed in S′ which are self-conjugate to S, and the like with Θ'.

3. If triangles can be inscribed in S, whose sides touch S′, then, if from any point P on S we draw tangents to S′ meeting S in Q, R, QR will touch S′. As P approaches A, a common point of S and S′, Q, R will tend to coincidence, and we shall have in the limit, if AB be a tangent to S′ at A, meeting S in B, BC will be a common tangent to S, S′, the triangle ABB being a triangle inscribed in S and circumscribing S′. We may then take $S = z^2+2fzx+2gxy$, $S' = y^2+2zx$, and the discriminant is

$$k^3g^2+(kf+1)^2 = 0, \text{ or } \Theta'^2 = 4\Delta'\Theta.$$

4. If quadrilaterals can be circumscribed to S′ such that the ends of two diagonals lie on S, we shall get, by similar considerations, that we can find A, B two common points of S, S′ such that the tangents to S′ at these points shall meet in a point C on S; the inscribed quadrangle being ACBCA. Thus $S = 2yz+2zx+2xy$, $S' = z^2+2hxy$, and the discriminant $2h^2(k+h)-(k+h)^2$. Thus $\Delta' = 2$, $\Theta = 2h-1$, $\Theta' = -2h$, $\Delta' = -h^2$.

Hence
$$\Theta'^2-4\Delta'\Theta = 4h^2(1+2h-1) = 8h^3,$$
$$\Theta'^3-4\Theta\Theta'\Delta' = -16h^4 = -16\Delta'^2 = -8\Delta'^2\Delta,$$
or, noticing the dimensions, we get the necessary condition as
$$\Theta'^3-4\Theta\Theta'\Delta'+8\Delta'^2\Delta = 0.$$

5. For a pentagon inscribed in S, whose sides taken in order shall touch S′, we get as the degenerate form ABCCBA, where A is a common point, AB the tangent to S′ meeting S in B, BC the second tangent to S′ from B; then the tangent to S at C must also touch S′. We may then take
$$S = 2x^2+2fyz-2xy, \quad S' = x^2+y^2+z^2-2yz-2zx-2xy,$$
the discriminant is
$$2k+1+2(k+1)(kf-1)-(2k+1)(kf-1)^2-1-(k+1)^2,$$
whence $\Delta = -2f^2$, $\Theta = -f^2+6f-1$, $\Theta' = 4f-4$, $\Delta' = -4$,
$$\Theta'^2-4\Delta'\Theta = 16[(f-1)^2-f^2+6f-1] = 64f,$$
$$\Theta'^3-4\Theta\Theta'\Delta' = 256f(f-1), \quad 8\Delta'^2\Delta = -256f^2,$$
$$\Theta^3-4\Theta\Theta'\Delta'+8\Delta'^2\Delta = -256f,$$
and the necessary relation is, when proper attention is paid to dimensions,
$$(\Theta'^2-4\Theta\Delta')^3 = 32\Delta'^2\Delta(\Theta'^3-4\Theta\Theta'\Delta'+8\Delta'^2\Delta).$$

[The true relation in such cases as this, where it might appear as if we had to select from several, can always be found as follows. Suppose the result we have found for Δ to be multiplied by λ^3, that for Θ by $\lambda^2\mu$, for Θ' by $\lambda\mu^2$, and for Δ' by μ^3, and eliminate the ratio $\lambda : \mu$. Thus above we should have
$$\Theta'^2-4\Theta\Delta' = 64f\lambda^2\mu^4,$$
$$\Theta'^3-4\Theta\Theta'\Delta'+8\Delta'^2\Delta = -256f\lambda^3\mu^6,$$
$$\Delta'^2\Delta = -32f^2\lambda^3\mu^6,$$
when it is obvious which is the true condition. This, however, can easily be done mentally.]

6. For a hexagon inscribed in S and whose sides in order touch S′, we shall have the limiting form ACDBDCA, where A, B are two common points, AC, BD tangents to S′ and chords of S; then CD must touch S′. We shall have
$$S = 2x^2+2fyz-2zx-2xy, \quad S' = x^2+y^2+z^2-2yz-2zx-2xy,$$
and the discriminant is
$$2k+1+2(k+1)^2(kf-1)-(2k+1)(kf-1)^2-2(k+1)^2,$$

so that $\Delta = 2f - 2f^2$, $\Theta = -f^2 + 8f - 4$, $\Theta' = 4f - 8$, $\Delta' = -4$,

whence $$\Theta'^2 - 4\Theta\Delta' = 16\{(f-2)^2 - f^2 + 8f - 4\} = 64f,$$

$$\Theta'^3 - 4\Theta\Theta'\Delta' = 256f(f-2);$$

$$\Delta'^2\Delta = 32f - 32f^2,$$

whence $$\Theta'^3 - 4\Theta\Theta'\Delta' + 8\Delta\Delta'^2 = -256f,$$

and $$\Theta'^3 - 4\Theta\Theta'\Delta' + 16\Delta\Delta'^2 = -256f^2,$$

and the condition is

$$(\Theta'^2 - 4\Theta\Delta')^3 = 4\{\Theta'^3 - 4\Theta\Theta'\Delta' + 8\Delta\Delta'^2\}\{\Theta'^3 - 4\Theta\Theta'\Delta' + 16\Delta\Delta'^2\}.$$

The general rule is now obvious: in order that polygons of $2n$ sides may be inscribed in S and circumscribed to S′, we must have a polygon $AA_1A_2 \ldots A_{n-1}B$, whose extreme points A, B are common points of S, S′, whose other points $A_1A_2 \ldots A_{n-1}$ lie on S and whose sides taken in order, AA_1, A_1A_2, $\ldots$ $A_{n-1}B$, touch S′. We shall not be at liberty to take any two of the four common points, but may take either of two pairs. So, for polygons of $2n+1$ sides, we must have a polygon $AA_1 \ldots A_n$, where A is any common point, $A_1A_2 \ldots A_n$ lie on S, and the sides taken in order AA_1, $A_1A_2 \ldots A_{n-1}A_n$, and the tangent to S at A_n must touch S′. It seems probable that the results can be more easily worked out in this manner than from the figures of finite area. I will conclude with—

7. The heptagon inscribed in S and whose sides touch S′. We shall have

$$S = 2x^2 + 2fyz + 2gzx - 2xy,$$

$$S' = x^2 + y^2 + z^2 - 2yz - 2zx - 2xy,$$

f, g being connected by the equation $fg + f + g = 0$. It will be found that

$$\Delta = -2f(f+g), \quad \Theta = -(f+g-1)^2 + 4f, \quad \Theta' = 4(f+g-1), \quad \Delta' = -4.$$

If we denote the conditions (in the forms already given) for in-and-escribed triangles, quadrangles, &c., by U_3, U_4, U_5, $\ldots$ we shall find in this case,

$$U_3 = \Theta'^2 - 4\Theta\Delta' = 64f, \qquad U_4 = -256f, \qquad U_5 = -4^3f^2g,$$

also $$\Delta'^2\Delta = 32f^2g.$$

We thus have $$\frac{U_4}{-4U_3} = 1, \text{ and } \frac{U_5}{\Delta'^2\Delta} = -2 \cdot 4^6,$$

so that the homogeneous condition is

$$\frac{U_5}{\Delta'^2\Delta} = -2 \cdot 4^6 \cdot \left(\frac{U_4}{-4U_3}\right)^3,$$

or $$U_5U_3^3 = 128\Delta'^2\Delta U_4^3,$$

or $$(\Theta'^2 - 4\Theta\Delta')^3\{(\Theta'^2 - 4\Theta\Delta')^3 - 32\Delta\Delta'^2(\Theta'^3 - 4\Theta\Theta'\Delta' + 8\Delta\Delta'^2)\}$$
$$= 128\Delta\Delta'^2(\Theta'^3 - 4\Theta\Theta'\Delta' + 8\Delta\Delta'^2)^3.$$

I do not remember where to look for the developed conditions of this kind to compare.

[Note.—(Added July, 1877.) When I wrote this paper I had not read Prof. Clifford's Note on the communication entitled "On the Transformation of Elliptic Functions," in which I find the method I have used is completely developed. (See "Transactions of the Mathematical Society," Vol. vii., page 230.) Two of the results, those of (1) and (3), communicated by me, were set in the Cambridge Mathematical Tripos Examination for 1876.]

January 11*th*, 1877.

SAMUEL ROBERTS, Esq., Treasurer, in the Chair.

Mr. G. W. Von Tunzelmann was elected a Member, and Colonel Petrie was present as a Visitor.

The following communications were made:—

"Determinant Conditions for Curves, or Surfaces, of the same order, having all their intersections common," Mr. J. Hammond; "Numerical Values of the first twelve powers of Π, of their reciprocals, and of certain other related quantities," Mr. J. W. L. Glaisher; "On some general classes of Multiple Definite Integrals," Mr. E. B. Elliott; "On the Partial Differential Equation $s+Pp+Qq+Z=0$," Prof. H. W. Lloyd Tanner; "On Methods for finding Equations to Axes of a Conic, and the Axes themselves, in Trilinear Coordinates," Mr. J. J. Walker; "On some Elliptic Function Properties," Prof. H. J. S. Smith.

The following presents were received:—

"Proceedings of the Royal Society," Vol. xxv., No. 176.

"Bulletin des Sciences Mathématiques et Astronomiques," Nov., Dec., 1876.

"Jahrbuch über die Fortschritte der Mathematik," siebenter Band, Jahrgang 1875, Heft i., Berlin, 1877.

"Sur les équations qui se rencontrent dans la Théorie de la Transformation des Fonctions elliptiques," par le Père Joubert, S. J., Paris, 1876; presented by Father Perry, S.J.

"Monatsbericht," September and October, 1876.

"Ueber das arithmetisch-geometrische Mittel aus vier Elementen," von C. W. Borchardt, Berlin, 1876.

"Archiv for Mathematik og Naturvidenskab udgivet af Sophus Lie, W. Müller og G. O. Sars," Ferste Bind, Ferste, Andet, Tredie Hefte, 1876.

"Allgemeine Theorie partielle Differential-Gleichungen 1·0, von Sophus Lie," Pts. 1, 2.

"Bidrag til Legemernes Molekylar-theorie af Cato M. Guldberg."

"Om Attractionen mellem to Cirkelflader af Ingenieur O. Pihl."

"Discussion aller Integrations-Methoden der partiellen Differential-Gleichungen 1·0 von Sophus Lie."

"Sur la Résolution des Equations du 2^{me}, 3^{me}, et 4^{me} degré par la fonction $\frac{n}{r}$ (X)," par Dr. Axel S. Guldberg (20 Dec. 1872).

(i.) "Zur Theorie des Integralitäts-faktors." (ii.) Veralgemeinerung und neue Verwerthung der Jacobischen Multiplicator-Theorie," von Sophus Lie.

The above from Det Kongelige Norske Universitat i Christiania.

Numerical Values of the First Twelve Powers of π, of their Reciprocals, and of certain other related quantities. *By* J. W. L. Glaisher, M.A., F.R.S.

[*Read January 11th*, 1877.]

1. I have so often wanted the first few powers of π to more figures than can be obtained by the use of seven-figure logarithms, that, some months since, I determined to have them calculated once for all to a sufficient extent to meet all cases that were likely to arise. My intention originally was merely to obtain the first twelve powers of π to twenty figures; but, when these were calculated, it seemed desirable to deduce from them also the values of their reciprocals.

2. The two following tables contain the values of π, π^2, π^3, ... π^{12} and of π^{-1}, π^{-2}, π^{-3} ... π^{-12} to twenty-two or more figures.

Table I.

n	π^n				
1	3·14159	26535	89793	23846	264
2	9·86960	44010	89358	61883	449
3	31·00627	66802	99820	17547	63
4	97·40909	10340	02437	23644	0
5	306·01968	47852	81453	2627	
6	961·38919	35753	04437	0302	
7	3020·29322	77767	92067	514	
8	9488·53101	60705	74007	129	
9	29809·09933	34462	11666	51	
10	93648·04747	60830	20973	72	
11	2 94204·01797	38905	97105	7	
12	9 24269·18152	33741	86222	6	

TABLE II.

n	π^{-n}					
1	0·31830	98861	83790	67153	7768	
2	0·10132	11836	42337	77144	3879	
3	0·03225	15344	33199	48918	4422	
4	0·01026	59822	54684	33518	9153	
5	0·00326	77636	43053	38547	2628	
6	0·00104	01614	73295	85229	6090	
7	0·00033	10936	80177	56676	43260	
8	0·00010	53903	91653	49366	63317	
9	0·00003	35468	03572	08869	12874	0
10	0·00001	06782	79226	86153	36620	4
11	0·00000	33990	01845	34103	10277	22
12	0·00000	10819	35890	52899	80492	70

The value of π^{-1} is well-known: it is given by Paucker (Grunert's "Archiv," t. i., p. 10) to 140 figures. All the other values in the tables were obtained by calculation, as explained in the next section. The values of π^2 and π^{-2} have been published before to more figures than my calculation extends, viz., in Maynard's list of constants* π^2 is given to 31 figures, and Paucker, in the paper just cited, has given the value of π^{-2} to 50 figures. I may mention also that in the same paper Paucker has given $\sqrt{\pi}$ to about 140 figures, and $\pi^{-\frac{1}{2}}$, $\pi^{\frac{1}{3}}$, $\pi^{\frac{2}{3}}$ as well as some other quantities involving π, to about 50 figures.

3. Table I. was calculated as follows:—The value of π to 24 figures was multiplied by itself, 24 figures of the result being retained: this value of π^2 was then multiplied by the value of π previously used, and 24 figures of the result retained, giving π^3: this was multiplied by π and so on up to π^{12}; there were thus eleven multiplications of 24 figures by 24 figures to 24 figures. The whole calculation was performed in duplicate, and the two calculations were brought into agreement.

Maynard's 31-figure value of π was then squared, and 31 figures of the square retained. This 31-figure value of π^4 was squared, and 31 figures of π^8 retained; π^8 was then multiplied twice by π^2 giving π^{10} and π^{12}. None of these values differed from those previously found by so much as a unit in the twenty-third figure, and accordingly the results given in the table (generally to 22 figures) should be correct to the last

* This list will be found in Templeton's "Millwright and Engineer's Pocket Companion" (1871), pp. 169—180.

figure. A small correction was applied to the values of π^2 and π^3, which was equivalent to the use of 25 figures of π.

Table II. was constructed in the first instance by dividing unity by each of the 24-figure values of π^2, π^3, ... π^{12} that were obtained by the duplicate 24-figure calculation. The divisions were performed to 24 significant figures. A second calculation was then made by dividing the value of π^{-2} by π, giving π^{-3}, which divided by π gave π^{-4} and so on to π^{-12}.

After these two calculations had been made completely accordant by the correction of a few slight errors, a third calculation was made (by another computer) as follows: the value of π^{-2} was divided by π^2, giving π^{-4}; this was divided by π^2, giving π^{-6}; and so on, to π^{-12}. For the verification of the uneven powers π^{-3} was divided by π^2 repeatedly till π^{-11} was reached. All the three calculations agreed to at least 23 figures.

The results in both Table I. and Table II. are, I believe, correct to the last figure (indeed in most cases it would have been quite safe to have given one figure more), but it is, of course, possible that any one of them may be in error by a unit in this figure. The last figure has been contracted, *i.e.*, it has been increased by unity, when the succeeding figures exceeded 50...

4. The results were verified by means of the equations

$$1+\pi+\pi^2 \ldots +\pi^{11} = \frac{\pi^{12}-1}{\pi-1},$$

$$1+\pi^{-1}+\pi^{-2} \ldots +\pi^{-11} = \frac{1-\pi^{-12}}{1-\pi^{-1}}.$$

The sum of the first eleven powers of π, increased by unity, was

4 31579·82447 03967 63120 124 ...,

while the value of $(\pi^{12}-1) \div (\pi-1)$, found by division, was

4 31579·82447 03967 63120 127 ...;

and the sum of the first eleven negative powers, increased by unity, was, to 25 decimal places,

1·46694 06197 86836 85681 05748,

which was exactly accordant to the same number of decimal places, with the value found by dividing $1-\pi^{-12}$ by $1-\pi^{-1}$. It must be noticed, however, that these verifications are not so perfect as at first sight they might seem to be, on account of the varying number of figures on the left of the decimal point in the one case and of ciphers on the right of the decimal point in the other. Thus although by the agreement of $1+\pi+\pi^2 \ldots +\pi^{11}$ and $(\pi^{12}-1) \div (\pi-1)$, π^{11} and π^{12} are verified directly to 22 figures, π^6 is only verified to 19, and π^2 to 17

figures. Similarly, π^{-12} and π^{-11} are only verified to 20 figures (25 places) although π^{-2} is verified to 25 figures.

5. M. Lefort* states that, in the formation of the "Tables du Cadastre," a table of the first twenty-six powers of $\frac{1}{2}\pi$ to 28 figures was constructed, and that it formed Table I. of Prony's Introduction. There can be little doubt that this table has not been published.

6. Most of the cases in which I have had occasion for the powers of π have arisen in the numerical verification of identities, in which the values of the terms had to be calculated to a certain number of decimal places (not figures). In calculating the value of a series

$$A_0 - A_1 x + A_2 x^2 - A_3 x^3 + \&c.$$

for a value of x greater than unity, the terms frequently increase very rapidly at first, and then only slowly diminish as the continual decrease of the coefficients overcomes the increase of the powers (the calculation of $\sin 3\pi$ from the series for $\sin x$ would be a case in point). Thus, in series where the integral portions of the terms cancel one another, a good many figures of the lower powers of x may be required in order to obtain the value of the series to only a moderate number of decimal places. In verifying certain elliptic function identities in which the terms on one side of the identity involve expressions such as $e^{-m\pi}$ or $e^{m\pi} \pm e^{-m\pi}$, it is frequently preferable to expand in powers of π, and to proceed by ordinary arithmetic, rather than to use ten-figure logarithms, even in cases where the latter afford a sufficient number of figures.

7. If we write $\quad S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \&c.,$

it is well known that

$$S_{2n} = \frac{1}{2} \frac{(2\pi)^{2n}}{1 \cdot 2 \cdot 3 \ldots 2n} B_n,$$

B_n being the n^{th} Bernoullian number, so that from the values of π^2, π^4, ... π^{12} may be deduced the values of S_2, S_4, ... S_{12}; the formulæ, in fact, being

$$S_2 = \frac{1}{6}\pi^2, \qquad S_8 = \frac{1}{9450}\pi^8,$$

$$S_4 = \frac{1}{90}\pi^4, \qquad S_{10} = \frac{1}{93555}\pi^{10},$$

$$S_6 = \frac{1}{945}\pi^6, \qquad S_{12} = \frac{691}{638512875}\pi^{12}.$$

The values in Table III. were obtained in this manner.

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* "Annales de l'Observatoire de Paris," t. iv. (1858). p. 135.

8. Writing $$s_n = \frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \&c.,$$

$$\sigma_n = \frac{1}{1^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \&c.;$$

then $$s_n = S_n - \frac{S_n}{2^{n-1}},$$

$$\sigma_n = S_n - \frac{S_n}{2^n}.$$

Tables IV. and V. were deduced from Table III. by means of these formulæ. In all three tables the last figure has been contracted (see end of § 3); it is liable to an error of a unit.

9. A table of the values of S_n from $n=1$ to $n=35$ to 16 decimal places was calculated by Legendre ("Traité des Fonctions Elliptiques," t. ii., p. 432). This table was an extension of one previously given by Euler, and there can be no doubt that it was obtained by Legendre directly from the series $1^{-n}+2^{-n}+3^{-n}+\&c.$ The values of s_n and σ_n (for the same values of n and also to 16 places), deduced as above from Legendre's Table of S_n, form Table IV. of my paper "On the Constants that occur in certain Summations by Bernoulli's Series." *

Legendre's values of S_n (as also the values of s_n and σ_n) are confirmed by those in the present paper up to the last figure of the former in the cases of $n = 2, 4, 6, 8$; for $n=10$ there is a unit-difference, viz. in the 16-place values the last three figures were, for S_n, 180; for s_n, 715; for σ_n, 447; while the extended values give 18085 ..., 71565 ..., 44825 .... For $n=12$, S_n and σ_n agree; but the ending of the 16-place value of s_n was 581, the extended value giving 58190 .... This is due to the accumulation of the errors produced by the contraction to 16 places; for the figures in the extended value are obtained by subtracting ... 49857 ... from ... 08048 ...; but the subtraction of ... 499 from ... 080 gives ... 581.

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* Proceedings of the London Mathematical Society, t. 4, p. 56 (1872).

Table III.

n	S_n				
2	1·64493	40668	48226	43647	2415
4	1·08232	32337	11138	19151	600
6	1·01734	30619	84449	13971	45
8	1·00407	73561	97944	33937	87
10	1·00099	45751	27818	08533	71
12	1·00024	60865	53308	04829	86

Table IV.

n	s_n				
2	0·82246	70334	24113	21823	6208
4	0·94703	28294	97245	91757	650
6	0·98555	10912	97435	10409	84
8	0·99623	30018	52647	89922	73
10	0·99903	95075	98271	56563	92
12	0·99975	76851	43858	19085	32

Table V.

n	σ_n				
2	1·23370	05501	36169	82735	4311
4	1·01467	80316	04192	05454	625
6	1·00144	70766	40942	12190	65
8	1·00015	51790	25296	11930	30
10	1·00001	70413	63044	82548	82
12	1·00000	18858	48583	11957	59

On some General Classes of Multiple Definite Integrals.

By E. B. Elliott, B.A., Queen's College, Oxford.

[*Read January 11th*, 1877.]

1. The methods of my former paper may be employed as follows to obtain some more general theorems including those before found as particular cases.

Let $S(p, q)$ be any symmetric function of p and q which does not become infinite for any positive values of p and q, or when either or both of them is zero or infinite. Then, using the fact that

$$\int_0^\infty \frac{\phi(ax)-\phi(a'x)}{x}\,dx = \log\frac{a}{a'}\{\phi(\infty)-\phi(0)\},$$

the double integral, in which a and b, a' and b' are positive constants,

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{S(ax, by)-S(a'x, b'y)}{xy}\,dx\,dy \\
&\quad= \int_0^\infty\int_0^\infty \frac{S(ax, by)-S(ax, b'y)}{xy}\,dx\,dy \\
&\qquad\qquad + \int_0^\infty\int_0^\infty \frac{S(ax, b'y)-S(a'x, b'y)}{xy}\,dx\,dy \\
&\quad= \log\frac{b}{b'}\int_0^\infty \frac{S(ax, \infty)-S(ax, 0)}{x}\,dx \\
&\qquad\qquad + \log\frac{a}{a'}\int_0^\infty \frac{S(\infty, b'y)-S(0, b'y)}{y}\,dy,
\end{aligned}$$

which if $\log\frac{a}{a'}+\log\frac{b}{b'}=0$, *i.e.*, if $ab=a'b'$, may be written

$$\begin{aligned}
&= \log\frac{a}{a'}\left\{\int_0^\infty \frac{S(ax, 0)-S(b'x, 0)}{x}\,dx \right. \\
&\qquad\qquad \left. -\int_0^\infty \frac{S(ax, \infty)-S(b'x, \infty)}{x}\,dx\right\} \\
&= \log\frac{a}{a'}\log\frac{a}{b'}\{S(\infty, 0)-S(0, 0)-S(\infty, \infty)+S(0, \infty)\} \\
&= -\log\frac{a}{a'}\log\frac{a}{b'}\{S(\infty, \infty)-2S(\infty, 0)+S(0, 0)\}\ldots\ldots(1).
\end{aligned}$$

In precisely the same way, or by transformation, if $S'(p, q)$ be a symmetric function not made infinite by any vanishing or negative, finite or infinite, values of p and q,

$$\int_{-\infty}^{0}\int_{-\infty}^{0} \frac{S'(ax, by) - S'(a'x, b'y)}{xy}\, dx\, dy$$
$$= -\log\frac{a}{a'}\log\frac{a}{b}\{S'(0, 0) - 2S'(0, -\infty) + S'(-\infty, -\infty)\} \ldots (2),$$

a, b, a', b' being positive constants connected by the same relation $ab = a'b'$.

In the same way each of the theorems proved below has a precisely corresponding one with regard to integrals between negative limits. These will be omitted for brevity, as they are easily written down.

Other general theorems may be obtained as in my former paper by the substitution of ϵ^x, ϵ^y, &c., for x, y, &c., and again by the substitution of $\log x$, $\log y$, &c., for x, y, &c.

2. Again $S(p, q, r)$ being any such symmetric function of p, q, r as not to be made infinite by any vanishing or positive values of its arguments, and a, b, c, a', b', c' being positive constants; then, introducing a seventh positive constant β, the triple integral

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{S(ax, by, cz) - S(a'x, b'y, c'z)}{xyz}\, dx\, dy\, dz$$
$$= \int_0^\infty\int_0^\infty\int_0^\infty \frac{S(ax, by, cz) - S(ax, \beta y, c'z)}{xyz}\, dx\, dy\, dz$$
$$+ \int_0^\infty\int_0^\infty\int_0^\infty \frac{S(ax, \beta y, c'z) - S(a'x, b'y, c'z)}{xyz}\, dx\, dy\, dz$$
$$= -\log\frac{b}{\beta}\log\frac{b}{c'}\int_0^\infty \frac{S(ax, \infty, \infty) - 2S(ax, \infty, 0) + S(ax, 0, 0)}{x}\, dx$$
$$-\log\frac{a}{a'}\log\frac{a}{b'}\int_0^\infty \frac{S(\infty, \infty, c'z) - 2S(\infty, 0, c'z) + S(0, 0, c'z)}{z}\, dz,$$

by (1), provided that $bc = \beta c'$ and $a\beta = a'b'$. If the additional condition holds, $\log\frac{a}{a'}\log\frac{a}{b'} + \log\frac{b}{\beta}\log\frac{b}{c'} = 0$, this may be written

$$= \log\frac{a}{a'}\log\frac{a}{b'}\left\{\int_0^\infty \frac{S(ax, \infty, \infty) - S(c'x, \infty, \infty)}{x}\, dx\right.$$
$$\left. -2\int_0^\infty \frac{S(ax, \infty, 0) - S(c'x, \infty, 0)}{x}\, dx + \int_0^\infty \frac{S(ax, 0, 0) - S(c'x, 0, 0)}{x}\, dx\right\}$$

$$= \log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\{S(\infty, \infty, \infty) - S(\infty, \infty, 0)$$
$$-2[S(\infty, \infty, 0) - S(0, \infty, 0)] + S(\infty, 0, 0) - S(0, 0, 0)\}$$

$$= \log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\{S(\infty, \infty, \infty) - 3S(\infty, \infty, 0)$$
$$+3S(\infty, 0, 0) - S(0, 0, 0)\} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The conditions in the constants under which this is true are, by elimination of β, easily written as the two independent ones $abc = a'b'c'$ and $\log\frac{a}{a'}\log\frac{a}{b'} = \log\frac{b}{c'}\log\frac{c}{c'}$. By a is of course denoted, in these results and conditions, any one of a, b, c, and by c' any one of a', b', c'. Notice that the second condition is the one by which the coefficient $\log\frac{\alpha}{\alpha'}\log\frac{\alpha}{b'}\log\frac{\alpha}{c'}$ is made a symmetric function of a, b, c, as it of course must be.

3. Again, the corresponding quadruple integral

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{S(ax, by, cz, dw) - S(a'x, b'y, c'z, d'w)}{xyzw}\,dx\,dy\,dz\,dw$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{S(ax, by, cz, dw) - S(ax, by, c'z, d'w)}{xyzw}\,dx\,dy\,dz\,dw$$

$$+ \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{S(ax, by, c'z, d'w) - S(a'x, b'y, c'z, d'w)}{xyzw}\,dx\,dy\,dz\,dw$$

$$= -\log\frac{c}{c'}\log\frac{c}{d'}\int_0^\infty\int_0^\infty \frac{S(ax, by, \infty, \infty) - 2S(ax, by, \infty, 0) + S(ax, by, 0, 0)}{xy}\,dx\,dy$$

$$-\log\frac{a}{a'}\log\frac{a}{b'}\int_0^\infty\int_0^\infty \frac{S(\infty, \infty, c'z, d'w) - 2S(\infty, 0, c'z, d'w) + S(0, 0, c'z, d'w)}{zw}\,dz\,dw,$$

by (1), provided that $cd = c'd'$ and $ab = a'b'$. If in addition we have $\log\frac{a}{a'}\log\frac{a}{b'} + \log\frac{c}{c'}\log\frac{c}{d'} = 0$, this may be written

$$= \log\frac{a}{a'}\log\frac{a}{b'}\left\{\int_0^\infty\int_0^\infty \frac{S(ax, by, \infty, \infty) - S(c'x, d'y, \infty, \infty)}{xy}\,dx\,dy - 2\int_0^\infty\int_0^\infty \frac{S(ax, by, \infty, 0) - S(c'x, d'y, \infty, 0)}{xy}\,dx\,dy\right.$$

$$\left. + \int_0^\infty\int_0^\infty \frac{S(ax, by, 0, 0) - S(c'x, d'y, 0, 0)}{xy}\,dx\,dy\right\}$$

$$= -\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\{S(\infty,\infty,\infty,\infty)-2S(\infty,0,\infty,\infty)+S(0,0,\infty,\infty)-2S(\infty,\infty,\infty,0)$$
$$+4S(\infty,0,\infty,0)-2S(0,0,\infty,0)+S(\infty,\infty,0,0)-2S(\infty,0,0,0)+S(0,0,0,0)\},$$

provided that also $ab = c'd'$,

$$= -\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\{S(\infty,\infty,\infty,\infty)-4S(\infty,\infty,\infty,0)+6S(\infty,\infty,0,0)-4S(\infty,0,0,0)+S(0,0,0,0)\}$$

Collected, the conditions for the truth of this are the four ..... (4).

$$ab = a'b' = cd = c'd', \qquad \log\frac{a}{a'}\log\frac{a}{b'} + \log\frac{c}{c'}\log\frac{c}{d'} = 0,$$

the first three of which are those by virtue of which the value is symmetric in a, b, c, d, as well as in a', b', c', d'.

4. The corresponding integral of the fifth order may now be considered.

By (3),
$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty \frac{S(ax, by, cz, du, ev)-S(ax, by, \gamma z, d'u, e'v)}{xyzuv}\,dx\,dy\,dz\,du\,dv$$
$$= \log\frac{c}{\gamma}\log\frac{c}{d'}\log\frac{c}{e'}\int_0^\infty\int_0^\infty \frac{S(ax, by, \infty, \infty, \infty)-3S(ax, by, \infty, \infty, 0)+3S(ax, by, \infty, 0, 0)-S(ax, by, 0, 0, 0)}{xy}\,dx\,dy,$$

provided that $cde = \gamma d'e'$ and $\log\frac{c}{\gamma}\log\frac{c}{d'} = \log\frac{d}{e'}\log\frac{e}{e'}$.

And in like manner
$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty \frac{S(ax, by, \gamma z, d'u, e'v)-S(a'x, b'y, c'z, d'u, e'v)}{xyzuv}\,dx\,dy\,dz\,du\,dv$$
$$= \log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\int_0^\infty\int_0^\infty \frac{S(\infty,\infty,\infty,d'u,e'v)-3S(\infty,\infty,0,d'u,e'v+3S(\infty,0,0,d'u,e'v)-S(0,0,0,d'u,e'v)}{uv}\,du\,dv,$$

provided that $ab\gamma = a'b'c'$ and $\log\frac{a}{a'}\log\frac{a}{b'} = \log\frac{b}{c'}\log\frac{\gamma}{c'}$.

If then also $\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}+\log\frac{c}{\gamma}\log\frac{c}{d'}\log\frac{c}{e'}=0$, by addition we have

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\,(ax,\,by,\,cz,\,du,\,ev)-\mathrm{S}\,(a'x,\,b'y,\,c'z,\,d'u,\,e'v)}{xyzuv}\,dx\,dy\,dz\,du\,dv$$

$$=-\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\left\{\int_0^\infty\int_0^\infty \frac{\mathrm{S}\,(ax,\,by,\,\infty,\,\infty,\,\infty)-3\mathrm{S}\,(ax,\,by,\,\infty,\,\infty,\,0)+3\mathrm{S}\,(ax,\,by,\,\infty,\,0,\,0)-\mathrm{S}\,(ax,\,by,\,0,\,0,\,0)}{xy}\,dx\,dy\right.$$

$$\left.-\int_0^\infty\int_0^\infty \frac{\mathrm{S}\,(d'x,\,e'y.\,\infty,\,\infty,\,\infty)-3\mathrm{S}\,(d'x,\,e'y,\,\infty,\,\infty,\,0)+3\mathrm{S}\,(d'x,\,e'y,\,\infty,\,0,\,0)-\mathrm{S}\,(d'x,\,e'y,\,0,\,0,\,0)}{xy}\,dx\,dy\right\}$$

$$=\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\left\{\mathrm{S}\,(\infty,\infty,\infty,\infty,\infty)-3\mathrm{S}\,(\infty,\infty,\infty,\infty,0)+3\mathrm{S}\,(\infty,\infty,\infty,0,0)-\mathrm{S}\,(\infty,\infty,0,0,0)\right.$$
$$-2\,[\mathrm{S}\,(\infty,0,\infty,\infty,\infty)-3\mathrm{S}\,(\infty,0,\infty,\infty,0)+3\mathrm{S}\,(\infty,0,\infty,0,0)-\mathrm{S}\,(\infty,0,0,0,0)]$$
$$\left.+\mathrm{S}\,(0,0,\infty,\infty,\infty)-3\mathrm{S}\,(0,0,\infty,\infty,0)+3\mathrm{S}\,(0,0,\infty,0,0)-\mathrm{S}\,(0,0,0,0,0)\right\}$$

by (1), provided that also $ab=d'e'$,

$$=\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\{\mathrm{S}\,(\infty,\infty,\infty,\infty,\infty)-5\mathrm{S}\,(\infty,\infty,\infty,\infty,0)+10\mathrm{S}\,(\infty,\infty,\infty,0,0)$$
$$-10\mathrm{S}\,(\infty,\infty,0,0,0)+5\mathrm{S}\,(\infty,0,0,0,0)-\mathrm{S}\,(0,0,0,0,0)\}\ldots\ldots\ldots(5).$$

By elimination of γ, the conditions in the ten constants for the truth of this may be written as the five

$$ab=d'e',\qquad cde=a'b'c',$$

$$\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}=-\log\frac{c}{e'}\log\frac{d}{e'}\log\frac{e}{e'}=\log\frac{a}{c'}\log\frac{b}{c'}\log\frac{a'b'}{ab}=-\log\frac{c}{d'}\log\frac{c}{e'}\log\frac{d'e'}{de}.$$

5. Again, the methods used in proving (8) and (9) below may both be applied to evaluate the sextuple integral

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{\mathrm{S}\,(ax,\ by,\ cz,\ du,\ ev,\ fw) - \mathrm{S}\,(a'x,\ b'y,\ c'z,\ d'u,\ e'v,\ f'w)}{xyzuvw}\, dx\, dy\, dz\, du\, dv\, dw.$$

It is thus found that, under the same seven conditions in the twelve constants, for which either of these results is true, the value of this integral is

$$-\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'}\{\mathrm{S}\,(\infty,\infty,\infty,\infty,\infty,\infty) - 6\mathrm{S}\,(\infty,\infty,\infty,\infty,\infty,0) + 15\,\mathrm{S}\,(\infty,\infty,\infty,\infty,0,0)$$
$$-20\mathrm{S}\,(\infty,\infty,\infty,0,0,0) + 15\mathrm{S}\,(\infty,\infty,0,0,0,0) - 6\mathrm{S}\,(\infty,0,0,0,0,0) + \mathrm{S}\,(0,0,0,0,0,0)\}\ldots\ldots(6).$$

Continuing in this way, conditions in the $2n$ involved constants $a_1, a_2, \ldots a_n,\ a'_1, a'_2, \ldots a'_n$ may be found for which the value of the corresponding n-tuple integral has a value which may be expressed symbolically as

$$(-1)^{n-1}\Pi_{r=1}^{r=n}\log\frac{a_1}{a'_r}\,\mathrm{S}\,(\infty - 0)^n.$$

6. Now, again, let $s\,(p, q)$ denote any symmetric function whatever of p and q, and $\mathrm{S}\,(s, s')$ any symmetric function of s and s', which does not become infinite when s and s' are any quantities whose values $s\,(p, q)$ may assume for positive or vanishing values of p and q; and let $a, b, a', b', c, d, c', d'$ be positive constants. Then the quadruple integral

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{S}\,[s\,(ax, by),\ s\,(cz, du)] - \mathrm{S}\,[s\,(a'x, b'y),\ s\,(c'z, d'u)]}{xyzu}\, dx\, dy\, dz\, du$$
$$= -\log\frac{c}{c'}\log\frac{c}{d'}\int_0^\infty\int_0^\infty \frac{\mathrm{S}\,[s\,(ax, by),\ s\,(\infty,\infty)] - 2\mathrm{S}\,[s\,(ax, by),\ s\,(\infty, 0)] + \mathrm{S}\,[s\,(ax, by),\ s\,(0, 0)]}{xy}\, dx\, dy$$
$$-\log\frac{a}{a'}\log\frac{a}{b'}\int_0^\infty\int_0^\infty \frac{\mathrm{S}\,[s\,(\infty,\infty),\ s\,(c'z, d'u)] - 2\mathrm{S}\,[s\,(\infty, 0),\ s\,(c'z, d'u)] + \mathrm{S}\,[s\,(0, 0),\ s\,(c'z, d'u)]}{zu}\, dz\, du,$$

by two applications of (1), provided that $ab = a'b'$ and $cd = c'd'$. If in addition $\log \frac{a}{a'} \log \frac{a}{b'} + \log \frac{c}{c'} \log \frac{c}{d'} = 0$, this may be written

$$\log \frac{a}{a'} \log \frac{a}{b'} \left\{ \int_0^\infty \int_0^\infty \frac{S[s(ax, by),\ s(\infty, \infty)] - S[s(c'x, d'y),\ s(\infty, \infty)]}{xy}\, dx\, dy \right.$$
$$- 2 \int_0^\infty \int_0^\infty \frac{S[s(ax, by),\ s(\infty, 0)] - S[s(c'x, d'y),\ s(\infty, 0)]}{xy}\, dx\, dy$$
$$\left. + \int_0^\infty \int_0^\infty \frac{S[s(ax, by),\ s(0, 0)] - S[s(c'x, d'y),\ s(0, 0)]}{xy}\, dx\, dy \right\}$$

$$= -\log \frac{a}{a'} \log \frac{a}{b'} \log \frac{a}{c'} \log \frac{a}{d'} \left\{ \begin{array}{l} S[s(\infty, \infty),\ s(\infty, \infty)] - 2S[s(\infty, 0),\ s(\infty, \infty)] + S[s(0, 0),\ s(\infty, \infty)] \\ -2S[s(\infty, \infty),\ s(\infty,\ 0)] + 4S[s(\infty, 0),\ s(\infty,\ 0)] - 2S[s(0, 0),\ s(\infty,\ 0)] \\ +\ S[s(\infty, \infty),\ s(\ 0,\ 0)] - 2S[s(\infty, 0),\ s(\ 0,\ 0)] + S[s(0, 0),\ s(\ 0,\ 0)] \end{array} \right\},$$

if the fourth condition $ab = c'd'$ be satisfied,

$$= -\log \frac{a}{a'} \log \frac{a}{b'} \log \frac{a}{c'} \log \frac{a}{d'} \left\{ \begin{array}{l} S[s(\infty, \infty),\ s(\infty, \infty)] - 4S[s(\infty, \infty),\ s(\infty, 0)] + 2S[s(\infty, \infty),\ s(0, 0)] \\ +4S[s(\infty,\ 0),\ s(\infty,\ 0)] - 4S[s(\infty,\ 0,\ s(0,\ 0)] + S[s(\ 0,\ 0),\ s(0, 0)] \end{array} \right\} (7).$$

7. Again, by two applications of (3), the sextuple integral, in which $s(p, q, r)$ is any symmetric function, and $S(s_1, s_2, s_3)$ a symmetric function not made infinite by any values of its arguments that $s(p, q, r)$ can assume for positive or vanishing values of p, q, r,

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty \frac{S\,[s\,(ax,\,by,\,cz),\,s\,(du,\,ev,\,fw)]-S\,[s\,(a'x,\,b'y,\,c'z),\,s\,(d'u,\,e'v,\,f'w)]}{xyzuvw}\,dx\,dy\,dz\,du\,dv\,dw$$

$$=\log\frac{d}{d'}\log\frac{d}{e'}\log\frac{d}{f'}\int_0^\infty\int_0^\infty\int_0^\infty \frac{\left\{\begin{array}{c} S\,[s\,(ax,\,by,\,cz),\,s\,(\infty,\,\infty,\,\infty)]-3S\,[s\,(ax,\,by,\,cz),\,s\,(\infty,\,\infty,\,0)]\\ +3S\,[s\,(ax,\,by,\,cz),\,s\,(\infty,\,0,\,0)]-S\,[s\,(ax,\,by,\,cz),\,s\,(0,\,0,\,0)]\end{array}\right\}}{xyz}\,dx\,dy\,dz$$

$$+\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\int_0^\infty\int_0^\infty\int_0^\infty \frac{\left\{\begin{array}{c} S\,[s\,(\infty,\,\infty,\,\infty),\,s\,(d'u,\,e'v,\,f'w)]-3S\,[s\,(\infty,\,\infty,\,0),\,s\,(d'u,\,e'v,\,f'w)]\\ +3S\,[s\,(\infty,\,0,\,0),\,s\,(d'u,\,e'v,\,f'w)]-S\,[s\,(0,\,0,\,0),\,s\,(d'u,\,e'v,\,f'w)]\end{array}\right\}}{uvw}\,du\,dv\,dw,$$

subject to the four conditions $def=d'e'f'$, $abc=a'b'c'$, $\log\frac{d}{d'}\log\frac{d}{e'}=\log\frac{e}{f'}\log\frac{f}{f'}$, $\log\frac{a}{a'}\log\frac{a}{b'}=\log\frac{b}{c'}\log\frac{c}{c'}$.

If, in addition, $\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}+\log\frac{d}{d'}\log\frac{d}{e'}\log\frac{d}{f'}=0$, this may be written

$$=-\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\left\{\int_0^\infty\int_0^\infty\int_0^\infty \frac{S\,[s(ax,\,by,\,cz),\,s(\infty,\,\infty,\,\infty)]-S\,[s(d'x,\,e'y,\,f'z),\,s(\infty,\,\infty,\,\infty)]}{xyz}\,dx\,dy\,dz\right.$$

$$-3\int_0^\infty\int_0^\infty\int_0^\infty \frac{S\,[s(ax,\,by,\,cz),\,s(\infty,\,\infty,\,0)]-S\,[s(d'x,\,e'y,\,f'z),\,s(\infty,\,\infty,\,0)]}{xyz}\,dx\,dy\,dz$$

$$+3\int_0^\infty\int_0^\infty\int_0^\infty \frac{S\,[s(ax,\,by,\,cz),\,s(\infty,\,0,\,0)]-S\,[s(d'x,\,e'y,\,f'z),\,s(\infty,\,0,\,0)]}{xyz}\,dx\,dy\,dz$$

$$\left.-\int_0^\infty\int_0^\infty\int_0^\infty \frac{S\,[s(ax,\,by,\,cz),\,s(0,\,0,\,0)]-S\,[s(d'x,\,e'y,\,f'z),\,s(0,\,0,\,0)]}{xyz}\,dx\,dy\,dz\right\},$$

which, if $abc=d'e'f'$, and $\log\frac{a}{d'}\log\frac{a}{e'}=\log\frac{b}{f'}\log\frac{c}{f'}$, becomes

$$= -\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'}$$
$$\times\Big\{\ \ \mathrm{S}[s(\infty,\infty,\infty),\ s(\infty,\infty,\infty)]-3\mathrm{S}[s(\infty,\infty,0),\ s(\infty,\infty,\infty)]+3\mathrm{S}[s(\infty,0,0),\ s(\infty,\infty,\infty)]-\ \mathrm{S}[s(0,0,0),\ s(\infty,\infty,\infty)]$$
$$-3\mathrm{S}[s(\infty,\infty,\infty),\ s(\infty,\infty,0)]+9\mathrm{S}[s(\infty,\infty,0),\ s(\infty,\infty,0)]-9\mathrm{S}[s(\infty,0,0),\ s(\infty,\infty,0)]+3\mathrm{S}[s(0,0,0),\ s(\infty,\infty,0)]$$
$$+3\mathrm{S}[s(\infty,\infty,\infty),\ s(\infty,0,0)]-9\mathrm{S}[s(\infty,\infty,0),\ s(\infty,0,0)]+9\mathrm{S}[s(\infty,0,0),\ s(\infty,0,0)]-3\mathrm{S}[s(0,0,0),\ s(\infty,0,0)]$$
$$-\ \mathrm{S}[s(\infty,\infty,\infty),\ s(0,0,0)]+3\mathrm{S}[s(\infty,\infty,0),\ s(0,0,0)]-3\mathrm{S}[s(\infty,0,0),\ s(0,0,0)]+\ \mathrm{S}[s(0,0,0),\ s(0,0,0)]\Big\}$$

$$= -\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'}$$
$$\times\Big\{\ \ \mathrm{S}[s(\infty,\infty,\infty),\ s(\infty,\infty,\infty)]-6\mathrm{S}[s(\infty,\infty,\infty),\ s(\infty,\infty,0)]+\ 6\mathrm{S}[s(\infty,\infty,\infty),\ s(\infty,0,0)]$$
$$+9\mathrm{S}[s(\infty,\infty,0),\ s(\infty,\infty,0)]-2\mathrm{S}[s(\infty,\infty,\infty),\ s(0,0,0)]-18\mathrm{S}[s(\infty,\infty,0),\ s(\infty,0,0)]$$
$$+6\mathrm{S}[s(\infty,\infty,0),\ s(0,0,0)]+9\mathrm{S}[s(\infty,0,0),\ s(\infty,0,0)]-\ 6\mathrm{S}[s(\infty,0,0),\ s(0,0,0)]$$
$$+\ \mathrm{S}[s(0,0,0),\ s(0,0,0)]\Big\}\quad\ldots\ldots\ldots\ldots\ (8).$$

Collected, the seven conditions in the twelve constants under which this is true may be written

$$abc = a'b'c' = def = d'e'f',$$

$$\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'} = \log\frac{a}{c'}\log\frac{b}{c'}\log\frac{c}{c'} = -\log\frac{d}{d'}\log\frac{d}{e'}\log\frac{d}{f'} = -\log\frac{d}{f'}\log\frac{e}{f'}\log\frac{f}{f'},$$

$$\log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'} = \log\frac{a}{f'}\log\frac{b}{f'}\log\frac{c}{f'}.$$

8. Lastly, by three applications of (1), the sextuple integral

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\,[s(ax,by),\,s(cz,du),\,s(ev,fw)]-\mathrm{S}\,[s(a'x,b'y),\,s(c'z,d'u),\,s(e'v,f'w)]}{xyzuvw}\,dx\,dy\,dz\,du\,dv\,dw$$

$$=-\log\frac{e}{e'}\log\frac{e}{f'}\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\,[s(ax,by),\,s(cz,du),\,s(\infty,\infty)]-2\mathrm{S}\,[s(ax,by),\,s(cz,du),\,s(\infty,0)]+\mathrm{S}\,[s(ax,by),\,s(cz,du),\,s(0,0)]}{xyzu}\times dx\,dy\,dz\,du$$

$$-\log\frac{c}{c'}\log\frac{c}{d'}\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\,[s(ax,by),\,s(\infty,\infty),\,s(e'v,f'w)]-2\mathrm{S}\,[s(ax,by),\,s(\infty,0),\,s(e'v,f'w)]+\mathrm{S}\,[s(ax,by),\,s(0,0),\,s(e'v,f'w)]}{xyvw}\times dx\,dy\,dv\,dw$$

$$-\log\frac{a}{a'}\log\frac{a}{b'}\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\,[s(\infty,\infty),\,s(c'z,d'u),\,s(e'v,f'w)]-2\mathrm{S}\,[s(\infty,0),\,s(c'z,d'u),\,s(e'v,f'w)]+\mathrm{S}\,[s(0,0),\,s(c'z,d'u),\,s(e'v,f'w)]}{zuvw}\times dz\,du\,dv\,dw,$$

provided that $ef=e'f'$, $cd=c'd'$, $ab=a'b'$. If, in addition, $\log\frac{c}{c'}\log\frac{c}{d'}=-\log\frac{a}{a'}\log\frac{a}{b'}-\log\frac{e}{e'}\log\frac{e}{f'}$, this may be written

$$=-\log\frac{e}{e'}\log\frac{e}{f'}\left\{\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\,[s(ax,by),\,s(cz,du),\,s(\infty,\infty)]-\mathrm{S}\,[s(ax,by),\,s(e'z,f'u),\,s(\infty,\infty)]}{xyzu}\,dx\,dy\,dz\,du\right.$$

$$-2\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\,[s(ax,by),\,s(cz,du),\,s(\infty,0)]-\mathrm{S}\,[s(ax,by),\,s(e'z,f'u),\,s(\infty,0)]}{xyzu}\,dx\,dy\,dz\,du$$

$$\left.+\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\,[s(ax,by),\,s(cz,du),\,s(0,0)]-\mathrm{S}\,[s(ax,by),\,s(e'z,f'u),\,s(0,0)]}{xyzu}\,dx\,dy\,dz\,du\right\}$$

$$+\log\frac{a}{a'}\log\frac{a}{b'}\left\{\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\,[s(ax,by),\,s(\infty,\infty),\,s(e'v,f'w)]-\mathrm{S}\,[s(c'x,d'y),\,s(\infty,\infty),\,s(e'v,f'w)]}{xyvw}\,dx\,dy\,dv\,dw\right.$$

$$-2\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\,[s(ax,by),\,s(\infty,0),\,s(e'v,f'w)]-\mathrm{S}\,[s(c'x,d'y),\,s(\infty,0),\,s(e'v,f'w)]}{xyvw}\,dx\,dy\,dv\,dw$$

$$\left.+\int_0^\infty\cdots\int_0^\infty \frac{\mathrm{S}\,[s(ax,by),\,s(0,0),\,s(e'v,f'w)]-\mathrm{S}\,[s(c'x,d'y),\,s(0,0),\,s(e'v,f'w)]}{xyvw}\,dx\,dy\,dv\,dw\right\};$$

and therefore, by (1), provided that also $cd = e'f'$, and $ab = c'd'$,

$$= \log\frac{e}{e'}\log\frac{e}{f'}\log\frac{c}{e'}\log\frac{c}{f'}$$

$$\times\left\{\int_0^\infty\int_0^\infty \frac{S[s(ax,by),s(\infty,\infty),s(\infty,\infty)]-2S[s(ax,by),s(\infty,0),s(\infty,\infty)]+S[s(ax,by),s(0,0),s(\infty,\infty)]}{xy}\,dx\,dy\right.$$

$$-2\int_0^\infty\int_0^\infty \frac{S[s(ax,by),s(\infty,\infty),s(\infty,0)]-2S[s(ax,by),s(\infty,0),s(\infty,0)]+S[s(ax,by),s(0,0),s(\infty,0)]}{xy}\,dx\,dy$$

$$\left.+\int_0^\infty\int_0^\infty \frac{S[s(ax,by),s(\infty,\infty),s(0,0)]-2S[s(ax,by),s(\infty,0),s(0,0)]+S[s(ax,by),s(0,0),s(0,0)]}{xy}\,dx\,dy\right\}$$

$$-\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}$$

$$\times\left\{\int_0^\infty\int_0^\infty \frac{S[s(\infty,\infty),s(\infty,\infty),s(e'v,f'w)]-2S[s(\infty,0),s(\infty,\infty),s(e'v,f'w)]+S[s(0,0),s(\infty,\infty),s(e'v,f'w)]}{vw}\,dv\,dw\right.$$

$$-2\int_0^\infty\int_0^\infty \frac{S[s(\infty,\infty),s(\infty,0),s(e'v,f'w)]-2S[s(\infty,0),s(\infty,0),s(e'v,f'w)]+S[s(0,0),s(\infty,0),s(e'v,f'w)]}{vw}\,dv\,dw$$

$$\left.+\int_0^\infty\int_0^\infty \frac{S[s(\infty,\infty),s(0,0),s(e'v,f'w)]-2S[s(\infty,0),s(0,0),s(e'v,f'w)]+S[s(0,0),s(0,0),s(e'v,f'w)]}{vw}\,dv\,dw\right\}.$$

If then, in addition, we have that $\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'} = \log\frac{e}{e'}\log\frac{e}{f'}\log\frac{c}{e'}\log\frac{c}{f'}$, by taking each term of the first bracket with the corresponding one of the second, and again applying (1), this becomes, since $ab = (c'd' = cd =)\ e'f'$,

$$= -\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'}$$

$$\times\Big\{\quad S[s(\infty,\infty),s(\infty,\infty),s(\infty,\infty)]-2S[s(\infty,0),s(\infty,\infty),s(\infty,\infty)]+\ S[s(0,0),s(\infty,\infty),s(\infty,\infty)]$$
$$-2S[s(\infty,\infty),s(\infty,0),s(\infty,\infty)]+4S[s(\infty,0),s(\infty,0),s(\infty,\infty)]-2S[s(0,0),s(\infty,0),s(\infty,\infty)]$$
$$+\ S[s(\infty,\infty),s(0,0),s(\infty,\infty)]-2S[s(\infty,0),s(0,0),s(\infty,\infty)]+\ S[s(0,0),s(0,0),s(\infty,\infty)]$$
$$-2S[s(\infty,\infty),s(\infty,\infty),s(\infty,0)]+4S[s(\infty,0),s(\infty,\infty),s(\infty,0)]-2S[s(0,0),s(\infty,\infty),s(\infty,0)]$$
$$+4S[s(\infty,\infty),s(\infty,0),s(\infty,0)]-8S[s(\infty,0),s(\infty,0),s(\infty,0)]+4S[s(0,0),s(\infty,0),s(\infty,0)]$$
$$-2S[s(\infty,\infty),s(0,0),s(\infty,0)]+4S[s(\infty,0),s(0,0),s(\infty,0)]-2S[s(0,0),s(0,0),s(\infty,0)]$$
$$+\ S[s(\infty,\infty),s(\infty,\infty),s(0,0)]-2S[s(\infty,0),s(\infty,\infty),s(0,0)]+\ S[s(0,0),s(\infty,\infty),s(0,0)]$$
$$-2S[s(\infty,\infty),s(\infty,0),s(0,0)]+4S[s(\infty,0),s(\infty,0),s(0,0)]-2S[s(0,0),s(\infty,0),s(0,0)]$$
$$+\ S[s(\infty,\infty),s(0,0),s(0,0)]-2S[s(\infty,0),s(0,0),s(0,0)]+\ S[s(0,0),s(0,0),s(0,0)]\Big\}$$

$$= -\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'}$$

$$\times\Big\{\quad S[s(\infty,\infty),s(\infty,\infty),s(\infty,\infty)]-\ 6S[s(\infty,\infty),s(\infty,\infty),s(\infty,0)]+3S[s(\infty,\infty),s(\infty,\infty),s(0,0)]$$
$$+12S[s(\infty,\infty),s(\infty,0),s(\infty,0)]-12S[s(\infty,\infty),s(\infty,0),s(0,0)]-8S[s(\infty,0),s(\infty,0),s(\infty,0)]$$
$$+12S[s(\infty,0),s(\infty,0),s(0,0)]+\ 3S[s(\infty,\infty),s(0,0),s(0,0)]-6S[s(\infty,0),s(0,0),s(0,0)]$$
$$+\ S[s(0,0),s(0,0),s(0,0)]\Big\}\quad\ldots\ldots(9).$$

Collected, the conditions for the twelve constants for the truth of this may be written as the seven

$$ab = cd = ef = a'b' = c'd' = e'f',$$

$$\log\frac{a}{a'}\log\frac{a}{b'} + \log\frac{c}{c'}\log\frac{c}{d'} + \log\frac{e}{e'}\log\frac{e}{f'} = 0,$$

$$\log\frac{a}{a'}\log\frac{a}{b'}\log\frac{a}{c'}\log\frac{a}{d'}\log\frac{a}{e'}\log\frac{a}{f'} + \log\frac{a}{f'}\log\frac{b}{f'}\log\frac{c}{f'}\log\frac{d}{f'}\log\frac{e}{f'}\log\frac{f}{f'} = 0.$$

[Addition.—The factors in the above results may be reduced to a very convenient symmetric form, suggested in a note on my former paper, for which I have to thank Professor Cayley.

Taking the factor $\log\frac{a_1}{a'_1}\log\frac{a_1}{a'_2}\log\frac{a_1}{a'_3}\ldots\log\frac{a_1}{a'_n}$ of the general formula in Art. 5, write a_1, a'_1, &c., for $\log a_1, \log a'_1$, &c. The factor becomes

$$(a_1-a'_1)(a_1-a'_2)\ldots\ldots(a_1-a'_n),$$

which, by the symmetry of the integral in the two sets of constants, must be capable of reduction to either of the other $2n-1$ forms,

$$\begin{array}{c}
(a_2-a'_1)(a_2-a'_2)\ldots\ldots(a_2-a'_n),\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
(a_n-a'_1)(a_n-a'_2)\ldots\ldots(a_n-a'_n),\\
-(a'_1-a_1)(a'_1-a_2)\ldots\ldots(a'_1-a_n),\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
-(a'_n-a_1)(a'_n-a_2)\ldots\ldots(a'_n-a_n).
\end{array}$$

Denote by A the common value of these products.

Now let $f(x)$ denote $(x-a'_1)(x-a'_2)\ldots(x-a'_n)$,

and $\phi(x)$ „ $(x-a_1)(x-a_2)\ldots(x-a_n)$.

Then every root of $\phi(x)=0$ inserted in $f(x)$ gives it the value A, and every root of $f(x)=0$ inserted in $\phi(x)$ gives it the value $-$A. From this it follows that

$$f(x) = A+\phi(x) \quad \text{for all values of } x.$$

Therefore, in particular, $A = f(0)-\phi(0)$

$$= (-1)^n\{a'_1a'_2\ldots a'_n - a_1a_2\ldots a_n\}.$$

Thus the values in Arts. 1—5 are all included in the form which may be symbolized by

$$\{\log a_1\log a_2\ldots\log a_n - \log a'_1\log a'_2\ldots\log a'_n\}\,S(\infty-0)^n.$$

The same form of factor applies also in the results of Arts. 6—8.]

On the Equation

$$\frac{d^2z}{dx\,dy}+\mathrm{P}\frac{dz}{dx}+\mathrm{Q}\frac{dz}{dy}+\mathrm{Z}=0.$$

By Prof. H. W. Lloyd Tanner, M.A.

[*Read January* 11*th*, 1877.]

1. The equation which forms the subject of this paper has not been discussed, as far as I know, except in the comparatively simple case in which P, Q do not involve the dependent variable z. An attempt is here made to supply this deficiency in part; viz., when the equation has a general solution of a kind which may be described as follows:— It expresses z in terms of two arbitrary functions $\phi(x)$, $\psi(y)$. These arbitrary functions may occur under the sign of indefinite integration; but the only other quantity under the integrating symbol is the differential of the argument of the arbitrary function. Thus we may have $\iint\ldots\phi(x)\,dx^n$, $\iint\ldots\psi(y)\,dy^n$ [written $\phi_n(x)$ and $\psi_n(y)$ in the sequel]; but not $\int[\phi(x)]^2\,dx$. Many integrals not included in this definition are easily reduced to the form required. For instance, $\int x\,\phi(x)\,dx$, which is $x\phi_1-\phi_2$. In Art. 13, below, another such integral occurs.

Our problem is then twofold: first, we must find the forms of P, Q, Z which are consistent with the assumed form of solution; secondly, we shall show how to find the solution of the equation when P, Q, Z are given functions of x, y, z included in the forms previously determined.

In order to get at these results, we write down the values of the differential coefficients of z arising from the assumed form of solution; substitute in the given equation; and express that the result vanishes independently of the forms of the arbitrary functions. We may do this in either of two ways. The coefficient of each power and product of the arbitrary functions may be equated to zero; or we may employ a method based upon the fact that if V be a function of ϕ, ψ, &c. which vanishes identically, then any and every differential coefficient of V with respect to ϕ, ψ, &c. also vanishes identically. A series of equations is thus obtained which will be found sufficient to answer the proposed questions.

It is found, as a result, that the forms of P, Q, Z must be such that the equation is reducible, by a change of the dependent variable only, to one or other of the following forms:—

$$\frac{d^2z}{dx\,dy}+\mathrm{P}e^z\frac{dz}{dx}+e^{-z}\frac{dz}{dy}+\frac{d\mathrm{P}}{dx}e^z=\mathrm{M}\quad\ldots\ldots\ldots\ldots\ldots(\alpha),$$

$$\frac{d^2z}{dx\,dy}+e^z\frac{dz}{dx}=\mathrm{M}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\beta),$$

$$\frac{d^2z}{dx\,dy} + z\frac{dz}{dx} + Zz + \frac{dZ}{dy} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (\gamma),$$

$$\frac{d^2z}{dx\,dy} + Ze^z = M \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\delta),$$

$$\frac{d^2z}{dx\,dy} + P\frac{dz}{dx} + Q\frac{dz}{dy} + Zz = M \quad \ldots\ldots\ldots\ldots (\epsilon).$$

In these forms, P, Q, Z, M are any functions of x, y. The last two cases are not discussed in this paper, as they have formed the subject of two papers by the writer published in the "Messenger of Mathematics."

I. *The forms of* P *and* Q.

2. The equation

$$\frac{d^2z}{dx\,dy} + P\frac{dz}{dx} + Q\frac{dz}{dy} + Z = 0 \quad \ldots\ldots\ldots\ldots\ldots (1),$$

where P, Q, Z are functions of x, y, z, is assumed to have a solution of the form

$$z = F\{x, y, \phi(x), \phi_1(x) \ldots\ldots \psi(y), \psi_1(y) \ldots\ldots\} \ldots\ldots\ldots (2),$$

and we seek the forms of P, Q consistent with this assumption.

From (2) we get, by differentiation,

$$\frac{dz}{dx} = \frac{dF}{dx} + \frac{dF}{d\phi}\phi' + \frac{dF}{d\phi_1}\phi + \ldots,$$

$$\frac{dz}{dy} = \frac{dF}{dy} + \frac{dF}{d\psi}\psi' + \frac{dF}{d\psi_1}\psi + \ldots,$$

$$\frac{d^2z}{dx\,dy} = \frac{d^2F}{dx\,dy} + \frac{d^2F}{d\phi\,d\psi}\phi'.\psi'$$
$$+ \left\{\frac{d^2F}{dy\,d\phi} + \frac{d^2F}{d\psi_1\,d\phi}\psi + \ldots\right\}\phi'$$
$$+ \left\{\frac{d^2F}{dx\,d\psi} + \frac{d^2F}{d\phi_1\,d\psi}\phi + \ldots\right\}\psi'$$
$$+ \text{terms not involving } \phi' \text{ or } \psi'.$$

We have, for the sake of shortness, omitted the arguments of the arbitrary functions.

On substituting these values in (1), it will be seen that the product $\phi'(x).\psi'(y)$ only occurs in $\frac{d^2z}{dx\,dy}$; and expressing that the coefficient of this product vanishes, we get an important result, viz.,

$$\frac{d^2F}{d\phi\,d\psi} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

3. The function $\phi'(x)$ occurs in $\frac{d^2z}{dx\,dy}$ and $\frac{dz}{dx}$ in the first power. Equating its coefficient to zero, we have

$$\frac{d^2F}{dy\,d\phi}+\frac{d^2F}{d\psi_1\,d\phi}\psi+\frac{d^2F}{d\psi_2\,d\phi}\psi_1+\ldots+P\frac{dF}{d\phi}=0 \quad\ldots\ldots\ldots\ldots(4).$$

Differentiate this with respect to ψ, remembering (3); then we have

$$\frac{d^2F}{d\psi_1\,d\phi}+\frac{dP}{d\psi}\cdot\frac{dF}{d\phi}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5);$$

and, repeating the differentiation,

$$\frac{d^2P}{d\psi^2}\cdot\frac{dF}{d\phi}=0.$$

Of this product the latter factor cannot vanish, so that

$$\frac{d^2P}{d\psi^2}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Hence P is linear in ψ, a result which is directly deducible from (4), since, by (3), $\frac{dF}{d\phi}$ does not involve ψ.

Now P, being a function of x, y, z, only involves ψ because it involves z. The equations (5), (6) may therefore be written

$$\frac{d^2F}{d\psi_1\,d\phi}+\frac{dP}{dz}\cdot\frac{dF}{d\phi}\cdot\frac{dF}{d\psi}=0 \quad\ldots\ldots\ldots\ldots\ldots(7),$$

$$\frac{d^2P}{dz^2}\left(\frac{dF}{d\psi}\right)^2+\frac{dP}{dz}\cdot\frac{d^2F}{d\psi^2}=0 \quad\ldots\ldots\ldots\ldots\ldots(8).$$

In (8) the function ϕ may occur in $\frac{d^2P}{dz^2}$, $\frac{dP}{dz}$, since these are, in the general case, functions of z. It does not occur in $\frac{dF}{d\psi}$ or $\frac{d^2F}{d\psi^2}$, because of (3). If then (8) be differentiated with respect to ϕ, and the result divided by the common factor $\frac{dF}{d\phi}$, which cannot vanish, we get

$$\frac{d^3P}{dz^3}\left(\frac{dF}{d\psi}\right)^2+\frac{d^2P}{dz^2}\cdot\frac{d^2F}{d\psi^2}=0.$$

Since $\frac{dF}{d\psi}$ cannot vanish, this equation is consistent with (8) if, and only if,

$$\begin{vmatrix}\frac{d^3P}{dz^3}, & \frac{d^2P}{dz^2}\\ \frac{d^2P}{dz^2}, & \frac{dP}{dz}\end{vmatrix}=0.$$

Hence, A being any function of x, y,

$$\frac{d^2P}{dz^2} = A\frac{dP}{dz}.$$

If A do not vanish, we have

$$P \equiv e^{Az+P_1} + P_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9);$$

if A be zero,

$$P \equiv P_1 z + P_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10),$$

where P_1, P_2 are functions of x, y only.

In just the same way, starting with the equation which expresses that the coefficient of ψ' vanishes, viz. with

$$\frac{d^2F}{dx\,d\psi} + \frac{d^2F}{d\phi_1\,d\psi}\varphi + \&c. + Q\frac{dF}{d\psi} = 0 \quad \ldots\ldots\ldots\ldots\ldots (4'),$$

we get equations for Q analogous to (5), &c.

II. *Case in which* $P \equiv e^{Az+P_1} + P_2$, *and* Q *involves* z.

4. This value of P, substituted in (8), gives

$$A\left(\frac{dF}{d\psi}\right)^2 + \frac{d^2F}{d\psi^2} = 0,$$

whence

$$\frac{dF}{d\psi} = \frac{1}{A}\cdot\frac{1}{\psi + F_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11).$$

Here F_1 does not involve ψ; and, by (3), it does not involve ϕ.

Thus (4′) becomes

$$-\frac{1}{A^2}\cdot\frac{dA}{dx}\cdot\frac{1}{\psi+F_1} - \frac{1}{A}\cdot\frac{1}{(\psi+F_1)^2}\cdot\frac{d.F_1}{dx} + \frac{Q}{A}\cdot\frac{1}{\psi+F_1} = 0 \ldots (12),$$

where $\frac{d.F_1}{dx}$ indicates the differential coefficient of F_1 with respect to x, no matter how involved.

Multiply this by $A^2(\psi+F_1)^2$, and differentiate with respect to ψ $\left(\text{which does not occur in A, } F_1, \text{ or } \frac{d.F_1}{dx}\right)$: thus,

$$-\frac{dA}{dx} + QA + \frac{dQ}{dz}\cdot\frac{dF}{d\psi}\cdot A(\psi+F_1) = 0,$$

or, by (11),

$$\frac{dQ}{dz} + AQ - \frac{dA}{dx} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13).$$

This is an equation which must be satisfied by Q when P is given by (9), in order that the equation (1) may have a solution of the assumed form (2); accordingly we must have

$$Q \equiv e^{-Az+Q_1} + \frac{1}{A}\cdot\frac{dA}{dx}.$$

Here e^{Q_1} may vanish, so that Q would not contain z. This is a case which will be discussed in the sequel. For the present we assume that this is not the case; so that Q is of the same form as P, but with $-A$ substituted for A. We may therefore infer

$$\frac{dF}{d\phi} = -\frac{1}{A}\cdot\frac{1}{\phi+F_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11'),$$

where F_2 contains neither ϕ nor ψ;

$$\frac{1}{A^2}\cdot\frac{dA}{dy}\cdot\frac{1}{\phi+F_2}+\frac{1}{A}\cdot\frac{1}{(\phi+F_2)^2}\cdot\frac{d.F_2}{dy}-\frac{P}{A}\cdot\frac{1}{\phi+F_2}=0 \ldots (12'),$$

and

$$P \equiv e^{Az+P_1}+\frac{1}{A}\cdot\frac{dA}{dy},$$

or these equations may be found directly from (4′), as (11), (12), (13) were found from (4).

5. The result at which we have arrived is that when $P \equiv e^{Az+P_1}+P_2$ and Q contains z, the equation (1) must be of the form

$$\frac{d^2z}{dx\,dy}+\left(e^{Az+P_1}+\frac{1}{A}\cdot\frac{dA}{dy}\right)\frac{dz}{dx}+\left(e^{-Az+Q_1}+\frac{1}{A}\cdot\frac{dA}{dx}\right)\frac{dz}{dy}+Z=0 \ldots (14),$$

where A, P_1, Q_1 contain x, y only. This form may be much simplified by a change of the dependent variable; viz., we take for new dependent variable ζ, defined by the equation

$$z=\frac{\zeta}{A}+h,$$

h being some function of x, y. From this we get

$$\frac{dz}{dx}=\frac{1}{A}\cdot\frac{d\zeta}{dx}-\frac{1}{A^2}\cdot\frac{dA}{dx}\cdot\zeta+\frac{dh}{dx},$$

$$\frac{d^2z}{dx\,dy}=\frac{1}{A}\cdot\frac{d^2\zeta}{dx\,dy}-\frac{1}{A^2}\cdot\frac{dA}{dy}\cdot\frac{d\zeta}{dx}-\frac{1}{A^2}\cdot\frac{dA}{dx}\cdot\frac{d\zeta}{dy}+\&c.$$

Introducing these values into (14), and multiplying by A so as to make the coefficient of $\frac{d^2\zeta}{dx\,dy}$ unity, we find for the coefficients of $\frac{d\zeta}{dx}$, $\frac{d\zeta}{dy}$,

$$e^{\zeta+Ah+P_1}, \quad e^{-\zeta-Ah+Q_1} \quad \text{respectively.}$$

If we define h by the condition

$$Ah=Q_1,$$

the transformed equation is of the form

$$\frac{d^2z}{dx\,dy}+P'e^z\frac{dz}{dy}+e^{-z}\frac{dz}{dy}+Z=0 \quad \ldots\ldots\ldots\ldots (15),$$

where $P'=e^{Ah+P_1}=e^{Q_1+P_1}$ is a function of x, y only. This is (14) when $A=1$, $Q_1=0$.

6. It only remains to determine the form of Z in the simplified equation (15).

From (11), (11′), putting A=1, and integrating, we get

$$z = \mathrm{F} \equiv \log(\psi+\mathrm{F}_1) - \log(\phi+\mathrm{F}_2) + \mathrm{F}_3 \quad \ldots\ldots\ldots\ldots \quad (16),$$

where F_3 contains neither ϕ nor ψ.

Again, in (12), put A=1, and

$$\mathrm{Q} = e^{-z} = \frac{\phi+\mathrm{F}_2}{\psi+\mathrm{F}_1}\, e^{-\mathrm{F}_3};$$

thus,
$$\frac{d\,.\,\mathrm{F}_1}{dx} = e^{-\mathrm{F}_3}(\phi+\mathrm{F}_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17).$$

So, in (12′), write A=1, and

$$\mathrm{P} = \mathrm{P}'e^z = \mathrm{P}'\,\frac{\psi+\mathrm{F}_1}{\phi+\mathrm{F}_2}\, e^{\mathrm{F}_3};$$

the result is
$$\frac{d\,.\,\mathrm{F}_2}{dy} = \mathrm{P}'e^{\mathrm{F}_3}(\psi+\mathrm{F}_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots (18).$$

By (16) we have

$$\frac{d^2z}{dx\,dy} = \frac{d}{dy}\cdot\frac{d}{dx}\cdot\log(\psi+\mathrm{F}_1) - \frac{d}{dx}\cdot\frac{d}{dy}\cdot\log(\phi+\mathrm{F}_2) + \frac{d^2\,.\,\mathrm{F}_3}{dx\,dy}$$

$$= \frac{d}{dy}\cdot\frac{\dfrac{d\,.\,\mathrm{F}_1}{dx}}{\psi+\mathrm{F}_1} - \frac{d}{dx}\cdot\frac{\dfrac{d\,.\,\mathrm{F}_2}{dy}}{\phi+\mathrm{F}_2} + \frac{d^2\,.\,\mathrm{F}_3}{dx\,dy}$$

$$= \frac{d}{dy}\cdot\left(e^{-\mathrm{F}_3}\frac{\phi+\mathrm{F}_2}{\psi+\mathrm{F}_1}\right) - \frac{d}{dx}\cdot\left(\mathrm{P}'e^{\mathrm{F}_3}\frac{\psi+\mathrm{F}_1}{\phi+\mathrm{F}_2}\right) + \frac{d^2\,.\,\mathrm{F}_3}{dx\,dy}, \text{ by (17), (18)}$$

$$= \frac{d}{dy}\,e^{-z} - \frac{d}{dx}(\mathrm{P}'e^z) + \frac{d^2\,.\,\mathrm{F}_3}{dx\,dy}, \text{ by (16)};$$

$$\therefore \quad \frac{d^2z}{dx\,dy} + \mathrm{P}'e^z\frac{dz}{dx} + e^{-z}\frac{dz}{dy} + \mathrm{Z}, = 0, = \frac{d^2\,.\,\mathrm{F}_3}{dx\,dy} - \frac{d\mathrm{P}'}{dx}\,e^z + \mathrm{Z} \ldots (15, 20).$$

This result will enable us to find the most general form of Z in (15). The equation we have just obtained,

$$\frac{d^2\,.\,\mathrm{F}_3}{dx\,dy} - \frac{d\mathrm{P}'}{dx}\,e^z + \mathrm{Z} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (20),$$

may involve ϕ, ψ in two ways. The last two terms involve ϕ, ψ only because they involve z; in other words, these two terms, considered as functions of ϕ, ψ, are some function of $\dfrac{\psi+\mathrm{F}_1}{\phi+\mathrm{F}_2}$.

On the other hand, F_3 does not involve ϕ, ψ; hence $\dfrac{d^2\,.\,\mathrm{F}_3}{dx\,dy}$ will be of the form

$$a\phi\psi + b\phi + c\psi + d,$$

where a, &c. (written for $\dfrac{d^2\,\mathrm{F}_3}{d\phi_1\,d\psi_1}$, &c.) do not involve ϕ, ψ.

Now this expression cannot be a function of $\frac{\psi+F_1}{\phi+F_2}$; therefore, in order that (20) may be true, we require that neither $\frac{d^2.F_3}{dx\,dy}$ nor $\frac{dP'}{dx}e^z-Z$ involve ϕ, ψ. Hence $\frac{dP'}{dx}e^z-Z$ is a function of x, y only; for, if it involved z, it would also involve ϕ, ψ. We may then put

$$\frac{dP'}{dx}e^z-Z=M,$$

or

$$Z=\frac{dP'}{dx}e^z-M \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (21),$$

where M is a function of x, y. Putting this general value of Z in (15), we get the equation (1) reduced to the form

$$\frac{d^2z}{dx\,dy}+P'e^z\frac{dz}{dx}+e^{-z}\frac{dz}{dy}+\frac{dP'}{dx}e^z=M \quad \ldots\ldots\ldots\ldots\ldots (\alpha).$$

7. The solution of this equation is now found without difficulty. Putting the value of Z given in (21) into (20), we get

$$\frac{d^2.F_3}{dx\,dy}=M,$$

or

$$F_3=\iint M\,dx\,dy,$$

which determines F_3 when M is given. When F_3 is thus determined, F_1 and F_2 are given by the equations (17), (18); and, these being found, the solution of (α) is (16); viz.,

$$e^z=\frac{\psi+F_1}{\phi+F_2}e^{F_3}.$$

The easiest way of finding F_1, F_2 in the general case is to assume

$$F_1=a_1\phi_1+a_2\phi_2+\ldots\ldots+b_1\psi_1+b_2\psi_2+\ldots\ldots,$$
$$F_2=\alpha_1\phi_1+\alpha_2\phi_2+\ldots\ldots+\beta_1\psi_1+\beta_2\psi_2+\ldots\ldots.$$

Substituting in (17), and equating to zero the coefficients of ϕ, ϕ_n, ψ_n, we get

$$a_1=e^{-F_3},$$
$$\frac{da_n}{dx}+a_{n+1}=e^{-F_3}\alpha_n,$$
$$\frac{db_n}{dx}=e^{-F_3}\beta_n.$$

Similarly, from (18),

$$\beta_1=P'e^{F_3},$$
$$\frac{d\alpha_n}{dy}=P'e^{F_3}a_n,$$
$$\frac{d\beta_n}{dy}+\beta_{n+1}=P'e^{F_3}b_n.$$

These equations are just sufficient to determine the coefficients a, b, α, β; as may be seen by putting them into the equivalent form

$$\alpha_1 = e^{-F_2}, \qquad \beta_1 = P' e^{F_2},$$

$$\alpha_{n+1} = -\frac{d\alpha_n}{dx} + e^{-F_2}\int P' e^{F_2}\alpha_n\, dy,$$

$$\beta_{n+1} = -\frac{d\beta_n}{dy} + P' e^{F_2}\int e^{-F_2}\beta_n\, dx,$$

(which give the α, β series); the a, b series then being found from

$$\frac{da_n}{dy} = P' e^{F_2}\alpha_n, \quad \frac{db_n}{dx} = e^{-F_2}\beta_n.$$

8. *Example:*

$$\frac{d^2z}{dx\,dy} + \frac{e^z}{(x+y)^2}\cdot\frac{dz}{dx} + e^{-z}\frac{dz}{dy} - \frac{2e^z}{(x+y)^3} + \frac{1}{(x+y)^2} = 0.$$

Here
$$F_2 = -\iint \frac{dx\,.\,dy}{(x+y)^2} = \log(x+y),$$

$$P' = \frac{1}{(x+y)^2};$$

$\therefore$
$$\alpha_1 = \frac{1}{x+y}, \quad \alpha_2 = \frac{1}{(x+y)^2} + \frac{1}{x+y}\cdot\int\frac{dy}{(x+y)^2} = 0;$$

$$\beta_1 = \frac{1}{x+y}, \quad \beta_2 = \frac{1}{(x+y)^2} + \frac{1}{x+y}\cdot\int\frac{dy}{(x+y)^2} = 0;$$

$$a_1 = \int\frac{dy}{(x+y)^2} = -\frac{1}{x+y}, \quad a_2 = 0;$$

$$b_1 = \int\frac{dx}{(x+y)^2} = -\frac{1}{x+y}, \quad b_2 = 0.$$

Thus,
$$F_1 = \frac{\phi_1}{x+y} - \frac{\psi_1}{x+y}, \quad F_2 = -\frac{\phi_1}{x+y} + \frac{\psi_1}{x+y};$$

and the required solution is

$$z = \log\left\{\frac{(x+y)\,\psi + \phi_1 - \psi_1}{(x+y)\,\phi - \phi_1 + \psi_1}(x+y)\right\}.$$

We may write this in the more usual form in which derivatives of the arbitrary functions are used, by taking $\phi_1(x)$, $\psi_1(y)$ as the original arbitrary functions. In this way the solution takes the form

$$z = \log\left\{\frac{\phi(x) - \psi(y) + (x+y)\,\psi'(y)}{\psi(y) - \phi(x) + (x+y)\,\phi'(x)}(x+y)\right\}.$$

III. *Case in which* $P \equiv e^{Az+P_1}+P_2$, *but* Q *does not involve* z.

9. A reference to (13) shows that, when Q does not involve z,

$$Q = \frac{1}{A} \cdot \frac{dA}{dx}.$$

The form which (1) assumes in this case is, then,

$$\frac{d^2z}{dx\,dy} + (e^{Az+P_1}+P_2)\frac{dz}{dx} + \frac{1}{A} \cdot \frac{dA}{dx} \cdot \frac{dz}{dy} + Z = 0.$$

This may be transformed as in Art. 5; and if we take h to be such that

$$Ah+P_1 = 0,$$

the transformed equation takes the form

$$\frac{d^2z}{dx\,dy} + (e^z+P_2)\frac{dz}{dx} + Z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (22).$$

Since $A=1$ in this form, (11) becomes

$$\frac{dF}{d\psi} = \frac{1}{\psi+F_1},$$

or
$$z = F = \log(\psi+F_1)+F_2 \ldots\ldots \ldots\ldots\ldots\ldots (23),$$

where F_2 does not contain ψ.

Also, since $Q=0$, (12) reduces to

$$\frac{d \cdot F_1}{dx} = 0.$$

showing that F_1 does not involve x in any way.

The equation (5) will now be useful. We notice that

$$\frac{dF}{d\phi} = \frac{dF_2}{d\phi},$$

and
$$P \equiv e^z+P_2 = (\psi+F_1)\,e^{F_2}+P_2,$$

so that
$$\frac{dP}{d\psi} = e^{F_2}.$$

Then (5) becomes
$$\frac{d^2F_2}{d\psi_1 d\phi} + e^{F_2}\frac{dF_2}{d\phi} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (24),$$

an equation which serves to determine the possible forms of F_2.

10. The following seems the easiest method of getting a general expression for F_2 which is of sufficiently simple form for our purpose. We observe that
$$e^{F_2} = \frac{1}{\phi+\psi_1}$$

is a solution of (24). Putting this value of F_2 in (23), we infer that

$$e^z = \frac{\psi(y)+F_1}{\phi(x)+\psi_1(y)}$$

is a solution of some form of (22): in fact, by trial we find that

$$e^z = \frac{\psi(y)}{\phi(x)+\psi_1(y)}$$

is a solution of $\dfrac{d^2z}{dxdy} + e^z\dfrac{dz}{dx} = 0$;

and it is the general solution, for it involves two independent arbitrary functions. But this differential equation is of the same form as (24), so that we can write down the general solution of the latter. To do this we recall that $\phi(x)$ is defined only by the equation $\dfrac{d\phi}{dy} = 0$, $\psi_1(y)$ by $\dfrac{d\psi_1}{dx} = 0$, and $\psi(y)$ by $\psi = \dfrac{d\psi_1}{dy}$. Hence the general value of F_2 is given by

$$e^{F_2} = \frac{1}{F_3+F_4}\cdot\frac{dF_4}{d\psi_1} \quad \text{.....(25),}$$

where

$$\frac{dF_3}{d\psi_1} = \frac{dF_4}{d\phi} = 0.$$

This result can be obtained, but with much more difficulty, by a direct integration of (24) as it stands; the result being reducible to the form (25).

11. We are now in a position to find the possible forms of P_2 in (22). For this purpose we substitute our new forms in (4). Since

$$\frac{dF}{d\phi} = \frac{dF_2}{d\phi} = -\frac{1}{F_3+F_4}\cdot\frac{dF_3}{d\phi}$$

and

$$P = \frac{\psi+F_1}{F_3+F_4}\cdot\frac{dF_4}{d\psi_1} + P_2,$$

(4) becomes

$$\frac{d}{dy}\cdot\left\{-\frac{1}{F_3+F_4}\cdot\frac{dF_3}{d\phi}\right\} - \left\{\frac{\psi+F_1}{F_3+F_4}\cdot\frac{dF_4}{d\psi_1}+P_2\right\}\frac{1}{F_3+F_4}\cdot\frac{dF_3}{d\phi} = 0,$$

or

$$-\frac{1}{F_3+F_4}\cdot\left\{\frac{d}{dy}\cdot\frac{dF_3}{d\phi}+P_2\frac{dF_3}{d\phi}\right\}$$
$$+\frac{1}{(F_3+F_4)^2}\left\{\frac{d\,.\,F_3}{dy}+\frac{d\,.\,F_4}{dy}-(\psi+F_1)\frac{dF_4}{d\psi_1}\right\}\frac{dF_3}{d\phi} = 0 \quad \text{...(26).}$$

Divide throughout by $\dfrac{1}{F_3+F_4}\cdot\dfrac{dF_3}{d\phi}$;

$$\therefore \quad \frac{d}{dy}\cdot\log\frac{dF_3}{d\phi}+P_2 = \frac{1}{F_3+F_4}\left\{\frac{d\,.\,F_3}{dy}+\frac{d\,.\,F_4}{dy}-(\psi+F_1)\frac{dF_4}{d\psi_1}\right\}.$$

Differentiate with respect to ϕ, noticing that the differentiations with respect to y, ϕ are independent and therefore commutative; then

$$\frac{d}{dy}\cdot\frac{d}{d\phi}\log\frac{dF_3}{d\phi}=-\frac{1}{(F_3+F_4)^2}\cdot\frac{dF_3}{d\phi}\left\{\frac{d\,.\,F_3}{dy}+\frac{d\,.\,F_4}{dy}-(\psi+F_1)\frac{dF_4}{d\psi_1}\right\}$$
$$+\frac{1}{F_3+F_4}\cdot\frac{d}{dy}\cdot\frac{dF_3}{d\phi}.$$

Replace the first term of the right member by its value given in (26), then

$$\frac{d}{dy}\cdot\frac{d}{d\phi}\log\frac{dF_3}{d\phi}=-\frac{1}{F_3+F_4}\left\{\frac{d}{dy}\cdot\frac{dF_3}{d\phi}+P_2\frac{dF_3}{d\phi}-\frac{d}{dy}\cdot\frac{dF_3}{d\phi}\right\}$$
$$=-\frac{1}{F_3+F_4}\cdot P_2\frac{dF_3}{d\phi}.$$

The left member of this equation is linear in ψ_1; for, F_3 not containing ψ_1, this function only occurs in the term

$$\psi_1\cdot\frac{d}{d\psi_2}\frac{d}{d\phi}\log\frac{dF_3}{d\phi}.$$

Unless P_2 vanishes, therefore, the expression

$$\frac{1}{F_3+F_4}$$

must also be linear in ψ_1. This is impossible; for suppose

$$F_3+F_4\equiv\frac{1}{A\psi_1+B},$$

where we only predicate of A, B that they do not contain ψ_1;

$$\therefore\qquad\frac{dF_4}{d\psi_1}=-\frac{A}{(A\psi_1+B)^2},$$

$$\therefore\qquad\frac{d^2F_4}{d\phi\,d\psi_1}(=0)=\frac{-(A\psi_1+B)\frac{dA}{d\phi}+2A\left\{\frac{dA}{d\phi}\psi_1+\frac{dB}{d\phi}\right\}}{(A\psi_1+B)^3},$$

$$\therefore\qquad A\frac{dA}{d\phi}\psi_1+2A\frac{dB}{d\phi}-B\frac{dA}{d\phi}=0,$$

which requires either $A=0$ or $\frac{dA}{d\phi}=\frac{dB}{d\phi}=0$. In the former case F_4 does not contain ψ_1, in the latter F_3 does not contain ϕ. Neither of these cases being admissible, we get

$$P_2=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27),$$

and

$$\frac{d}{dy}\cdot\frac{d}{d\phi}\log\frac{dF_3}{d\phi}=0.$$

12. This last result enables us to simplify (25) very materially. It gives by integration

$$F_3 \equiv \Psi_1 \times (\Phi + \Psi_2),$$

where $\frac{d \cdot \Phi}{dy} = 0$, and Ψ_1, Ψ_2 do not contain ϕ. Also since $\frac{dF_3}{d\psi_1} = 0$, neither Ψ_1 nor Ψ_2 contains ψ_1. If now we write (25) in the form

$$\frac{1}{\Phi + \Psi_2 + \frac{F_4}{\Psi_1}} \cdot \frac{1}{\Psi_1} \cdot \frac{dF_4}{d\psi_1},$$

and put

$$F_4' \equiv \Psi_2 + \frac{F_4}{\Psi_1},$$

so that

$$\frac{dF_4'}{d\psi_1} = \frac{1}{\Psi_1} \cdot \frac{dF_4}{d\psi_1}, \quad \frac{dF_4'}{d\phi} = 0,$$

we get

$$e^{F_3} = \frac{1}{\Phi + F_4'} \cdot \frac{dF_4'}{d\phi_1},$$

where Φ includes neither y nor any function of y; and F_4' does not involve ϕ.

Using this form, (26) and (23) become

$$\frac{d \cdot F_4}{dy} - (\psi + F_1) \frac{dF_4}{d\psi_1} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (28),$$

and

$$z = \log(\psi + F_1) - \log(\Phi + F_4) + \log \frac{dF_4}{d\psi_1} \ldots\ldots\ldots (29),$$

$$\therefore \quad \frac{dz}{dy} = \frac{\psi' + \frac{d \cdot F_1}{dy}}{\psi + F_1} - \frac{1}{\Phi + F_4} \cdot \frac{d \cdot F_4}{dy} + \frac{d}{dy} \cdot \log \frac{dF_4}{d\psi_1},$$

$$\therefore \quad \frac{dz}{dy} + e^z = \frac{\psi' + \frac{d \cdot F_1}{dy}}{\psi + F_1} + \frac{d}{dy} \cdot \log \frac{dF_4}{d\psi_1}, \text{ by (28)},$$

$$\therefore \quad \frac{d^2z}{dx\,dy} + e^z \frac{dz}{dx} = \frac{d}{dx} \cdot \frac{d}{dy} \cdot \log \frac{dF_4}{d\psi_1} = -Z, \text{ by (22) and (27)} \ldots (30).$$

Now F_4 does not contain ϕ, ψ so that the second member is of the form

$$a\phi\psi + b\phi + c\psi + d,$$

and the same must be true of Z; but no function of z can be of this form since z involves ϕ, ψ, as being a function of

$$\frac{\psi + F_1}{\Phi + F_3}.$$

Hence Z is a function of x, y only; and (22) is thus reduced to the form

$$\frac{d^2z}{dx\,dy} + e^z \frac{dz}{dx} = M \ldots\ldots\ldots\ldots\ldots\ldots (\beta),$$

where M has been written for $-Z$.

13. The solution is now easily found. By (30),

$$\frac{dF_4}{d\psi_1} = e^{\iint M dx\, dy} = M' \text{ say.}$$

No generality is gained by keeping F_1, so we shall suppose it to vanish; then (28) becomes

$$\frac{dF_4}{dy} = M' \,.\, \psi(y),$$

$$\therefore \qquad F_4 = \int M' \,.\, \psi(y) \,.\, dy.$$

Lastly, $\Phi\{x, \phi(x) \ldots\ldots\}$ is no more general than $\phi(x)$; we may therefore put

$$\Phi = \phi(x).$$

Then (29) becomes

$$z = \log \frac{M' \,.\, \psi(y)}{\phi(x) + \int M' \psi(y) \,.\, dy},$$

which, when

$$M' = e^{\iint M dx\, dy},$$

is the general solution of (β).

This solution may be reduced to the form (2) by integrating $\int M' \psi(y)\, dy$ by parts: but it seems more convenient to leave it as it stands.

IV. *Case in which* $P \equiv P_1 z + P_2$.

14. In this case (8) becomes

$$P_1 \,.\, \frac{d^2F}{d\psi^2} = 0,$$

or, since we suppose that P_1 does not vanish,

$$\frac{d^2F}{d\psi^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (31).$$

Hence, differentiating (4′) with respect to ψ, we have

$$\frac{dQ}{d\psi} \,.\, \frac{dF}{d\psi} = 0.$$

The last factor of this product cannot vanish: so that $\frac{dQ}{d\psi}$, and therefore $\frac{dQ}{dz}$, must vanish. Hence (1) becomes

$$\frac{d^2z}{dx\, dy} + (P_1 z + P_2)\frac{dz}{dx} + Q\frac{dz}{dy} + Z = 0,$$

where P_1, P_2, Q are functions of x, y only.

This may be simplified by taking ζ for new dependent variable, ζ being defined by $z = h\zeta + k$.

Here h, k are two functions of x, y: and when they are properly chosen the transformed equation is of the form

$$\frac{d^2z}{dx\,dy} + P_1 z \frac{dz}{dx} + Z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (32),$$

where the P_1, Z are not the same as before. Moreover, we can choose h in an infinite number of ways to get this result: and can make use of this freedom to replace P_1 by $P_1 Y$, where Y is any function we please of y only. This will be of use in the sequel.

Since $Q = 0$, (4′) becomes

$$\frac{d}{dx} \cdot \frac{dF}{d\psi} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (33),$$

or

$$\frac{d}{d\psi} \frac{d}{dx} . F = 0,$$

since x, ψ are independent. This means that $\frac{dz}{dx}$ does not involve ψ: and consequently $\frac{d^2z}{dx\,dy}$ can only involve it linearly. Therefore Z is linear in ψ, by (32): but z also being linear in ψ, by (31), we infer that Z is linear in z: that is, the equation becomes

$$\frac{d^2z}{dx\,dy} + P_1 z \frac{dz}{dx} + Z_1 z = M \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (34),$$

15. We shall now show that P_1 is a function of y only; so that, in consequence of the remark made upon (32), we may suppose it reduced to unity. The equation (4) becomes in this case

$$\frac{d}{dy} \cdot \frac{dF}{d\phi} + P_1 F \frac{dF}{d\phi} = 0,$$

or

$$\frac{d}{dy} . \log \frac{dF}{d\phi} + P_1 F = 0.$$

Differentiating this with respect to ϕ, we get

$$\frac{d}{dy} \cdot \frac{d}{d\phi} \log \frac{dF}{d\phi} + P_1 \frac{dF}{d\phi} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (35).$$

Equating to zero the coefficient of ψ in this identity, we get

$$\frac{d}{d\psi_1} \frac{d}{d\phi} \log \frac{dF}{d\phi} = 0,$$

which shows that $\frac{d}{d\phi} \log \frac{dF}{d\phi}$ does not contain ψ_1. The first term of

(35) is therefore linear in ψ, and the same must be true of $\frac{d\mathrm{F}}{d\phi}$ on account of (35);

$$\therefore \quad \frac{d\mathrm{F}}{d\phi} = \frac{d\mathrm{F}_1}{d\phi} \cdot \{\psi_1 + \mathrm{F}_2\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (36),$$

where $\frac{d\mathrm{F}_1}{d\phi}$, F_2 contain neither ψ_1 nor ψ.

Substitute this in (7) and reduce,

$$\therefore \quad \frac{d\mathrm{F}}{d\psi} = -\frac{1}{\mathrm{P}_1} \cdot \frac{1}{\psi_1 + \mathrm{F}_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (37).$$

Hence, by (33),

$$\frac{d\mathrm{P}_1}{dx}(\psi_1+\mathrm{F}_2) + \mathrm{P}_1 \frac{d\,.\,\mathrm{F}_2}{dx} = 0.$$

Since neither F_2 nor $\frac{d\,.\,\mathrm{F}_2}{dx}$ contains ψ_1, this requires

$$\frac{d\mathrm{P}_1}{dx} = 0, \quad \frac{d\,.\,\mathrm{F}_2}{dx} = 0.$$

Therefore P_1 is a function of y only, and (34) can now be reduced to the form

$$\frac{d^2z}{dx\,dy} + z\frac{dz}{dx} + \mathrm{Z}z = \mathrm{M} \quad \ldots\ldots\ldots\ldots\ldots\ldots (38).$$

At this point unexpected difficulties have arisen which have induced me to complete the investigation indirectly. If z be expressible in the form (1), then $\log\left(\frac{dz}{dx} + \mathrm{Z}\right)$ is also expressible in this form. I form then an equation (by differentiating (38) with respect to x, and eliminating z), in which $\log\left(\frac{dz}{dx}+\mathrm{Z}\right)$ is dependent variable. This equation is

$$\frac{d^2\pi}{dx\,dy} + \left\{\frac{d\mathrm{Z}}{dy} + \mathrm{M}\right\} e^{-\pi}\frac{d\pi}{dx} + e^{\pi} - \mathrm{Z} - \left\{\frac{d^2\mathrm{Z}}{dx\,dy} + \frac{d\mathrm{M}}{dx}\right\} e^{-\pi} = 0,$$

where π is written for $\log\left(\frac{dz}{dx} + \mathrm{Z}\right)$.

Now this cannot be solved in the form (2) unless

$$\frac{d\mathrm{Z}}{dy} + \mathrm{M} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (39),$$

when it becomes

$$\frac{d^2\pi}{dx\,dy} + e^{\pi} - \mathrm{Z} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (40).$$

By (39), (38) reduces to

$$\frac{d^2z}{dx\,dy} + z\frac{dz}{dx} + \mathrm{Z}z + \frac{d\mathrm{Z}}{dy} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (\gamma).$$

The solution of (γ) can at once be found by a simple integration when that of (40) is known. But (40) has only been solved in the case

in which Z vanishes. ("Messenger of Mathematics," Vol. v., p. 156.) In this case (γ) reduces to

$$\frac{d^2z}{dxdy} + z\frac{dz}{dx} = 0,$$

the solution of which is known to be

$$z = 2\frac{\psi_1(y)}{\phi(x)+\psi_2(y)} - \frac{\psi(y)}{\psi_1(y)}$$

("Messenger of Mathematics," Vol. vi., p. 41).

On the Quaternion Forms of some General Propositions in Fluid Motion. *By* Mr. J. G. Butcher, B.A., Fellow of Trinity College, Cambridge.

[*Read December 14th*, 1876.]

1. I propose here to prove some general propositions on the pure kinematics of a continuous medium by Quaternion methods.

Let ∇ denote, as usual, Hamilton's operator $i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}$. Then, if σ be any vector whatever, π a small vector displacement of a point (x, y, z), it is easily proved, as in Tait's "Quaternions," p. 216,

$$(S\pi\nabla)\,\sigma = -D_\pi\sigma,$$

where D_π denotes total differentiation in virtue of having passed from one end to the other of the vector π.

Thus $S\pi\nabla$ is a scalar operation. If we perform it on a scalar function P, we can write $(S\pi\nabla)\,P$ or $S\,.\,\pi\nabla P$ indifferently; for

$$\begin{aligned} S\,.\,\pi\nabla P &= S\,.\,(S\pi\nabla + \nabla\pi\nabla)\,P \\ &= (S\pi\nabla)\,P. \end{aligned}$$

But of course $(S\pi\nabla)\,\sigma$ is not equal to $S\,.\,\pi\nabla\sigma$.

Let us consider a point P whose vector velocity is σ. Take a point Q in the vicinity of P whose vector distance from P is π.

After a small time δt, P will have come to P', Q to Q'; and if σ_1 be the vector velocity of Q, we have

$$\sigma_1 = \sigma - (S\pi\nabla)\,\sigma.$$

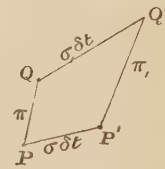
Now

$$\pi - \sigma\delta t = \pi_1 - \sigma_1\delta t.$$

Hence

$$\pi_1 - \pi = -(S\pi\nabla)\,\sigma\delta t;$$

and we see that π_1 is a linear and vector function of π. Call it $\phi\pi$, and



ϕ is what we may call the "deformation-function" of the group of points round P.

Decomposing ϕ by the usual methods, we have

$$\begin{aligned}\phi\pi &= \pi - (\mathrm{S}\pi\nabla)\,\sigma\delta t \\ &= \bar{\phi}\pi + \mathrm{V}\epsilon\pi,\end{aligned}$$

where $\bar{\phi}$ denotes a pure strain (a self-conjugate function), and $\mathrm{V}\epsilon\pi$ denotes a rotation of amount $\mathrm{T}\epsilon$.

To find the values of $\bar{\phi}$ and ϵ, we may use a method of Tait's (*Edin. Trans.*, Vol. xxvi., p. 79), and put

$$\phi = \frac{\phi+\phi'}{2} + \frac{\phi-\phi'}{2},$$

where ϕ' is the conjugate function to ϕ; and then we know that $\frac{\phi+\phi'}{2}$ represents the self-conjugate part, $\frac{\phi-\phi'}{2}$ the rotational part, of ϕ.

To find ϕ', we have

$$\begin{aligned}\mathrm{S}\rho\phi\pi &= \mathrm{S}\rho\,[\pi - (\mathrm{S}\pi\nabla)\,\sigma] \\ &= \mathrm{S}\rho\pi - (\mathrm{S}\pi\nabla)\,\mathrm{S}\rho\sigma, \quad \nabla \text{ operating on } \sigma \text{ only}, \\ &= \mathrm{S}\pi\rho - \mathrm{S}\,.\,\pi\nabla\mathrm{S}\rho\sigma \\ &= \mathrm{S}\pi\,(\rho - \nabla\,.\,\mathrm{S}\rho\sigma) \\ &= \mathrm{S}\pi\phi'\rho\,;\end{aligned}$$

so that

$$\phi'\rho = \rho - \nabla\,.\,\mathrm{S}\rho\sigma.$$

Hence

$$\begin{aligned}\phi\pi &= \pi - \frac{\delta t}{2}\,[(\mathrm{S}\pi\nabla)\,\sigma + \nabla\,.\,\mathrm{S}\pi\sigma] - \frac{\delta t}{2}\,[(\mathrm{S}\pi\nabla)\,\sigma - \nabla\,.\,\mathrm{S}\pi\sigma] \\ &= \pi - \frac{\delta t}{2}\,[(\mathrm{S}\pi\nabla)\,\sigma + \nabla\,.\,\mathrm{S}\pi\sigma] - \frac{\delta t}{2}\,\nabla\,.\,\pi\nabla\,.\,\nabla\sigma \\ &= \bar{\phi}\pi + \mathrm{V}\epsilon\pi,\end{aligned}$$

where

$$\bar{\phi}\pi = \pi - \frac{\delta t}{2}\,[(\mathrm{S}\pi\nabla)\,\sigma + \nabla\,.\,\mathrm{S}\pi\sigma],$$

$$\epsilon = \frac{\delta t}{2}\,\nabla\,.\,\nabla\sigma.$$

Thus ϵ is the vector axis of rotation of the group of points round P; or, in Cartesians, if $\sigma = iu + jv + kw$,

$$\epsilon = \tfrac{1}{2}\delta t\left\{ i\left(\frac{dw}{dy} - \frac{dv}{dz}\right) + j\left(\frac{du}{dz} - \frac{dw}{dx}\right) + k\left(\frac{dv}{dx} - \frac{du}{dy}\right)\right\},$$

the well-known results of Stokes and Helmholtz.

2. Thus the motion consists of two parts:

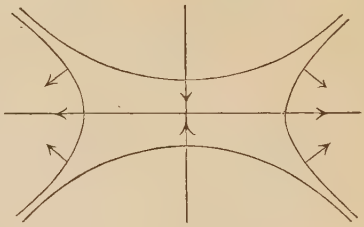
(1) A pure strain, given by $\bar{\phi}\pi$.

If we construct the quadric $\mathrm{S}\pi\bar{\phi}\pi = \text{const.}$, then $\bar{\phi}\pi$ is the direction of the normal at the extremity of π, and the instantaneous motion of

the point Q due to this cause alone is in the direction of the normal to this quadric, and its velocity

$$\propto \frac{1}{\mathrm{T}\phi\pi} \propto \frac{1}{\text{perp. on tang. plane at Q}}.$$

We may sketch the motion roughly thus, where the arrow-heads indicate the directions of motion.



(2) A simple rotation round a line instantaneously fixed at P, which line is the vector $\mathrm{V}.\nabla\sigma$, the amount of rotation being $\frac{\delta t}{2}\mathrm{TV}.\nabla\sigma$.

We may examine two particular cases.

Case I.—*Motion Non-rotational.*

The criterion for this is $\mathrm{V}.\nabla\sigma = 0$, and in this case, if ρ be the vector to P, we must have (Tait's "Quaternions," p. 174) $\mathrm{S}.\sigma d\rho$ a perfect differential $= -d.\mathrm{F}(\rho)$ say, where F is a scalar function.

But
$$d\mathrm{F} = \frac{d\mathrm{F}}{dx}dx + \frac{d\mathrm{F}}{dy}dy + \frac{d\mathrm{F}}{dz}dz,$$
$$= -\mathrm{S}d\rho\nabla\mathrm{F},$$

so that we must put
$$\sigma = \nabla\mathrm{F}(\rho),$$

and F is what is ordinarily called the "velocity-potential." If the fluid be in addition incompressible, we have of course

$$\nabla^2\mathrm{F} = 0.$$

We may remark in passing, that if

$$\mathrm{F}\rho = \mathrm{C}$$

be the equation of one of a family of surfaces, and

$$d.\mathrm{F}\rho = 0, \quad \text{or} \quad \mathrm{S}d\rho\nabla\mathrm{F} = 0,$$

be the differential equation of the family, then for a consecutive surface

$$\mathrm{F}\rho + \delta.\mathrm{F}\rho = \mathrm{F}(\rho) - \mathrm{S}\delta\rho\nabla\mathrm{F} = \mathrm{F}\rho - \mathrm{S}\sigma\delta\rho = \mathrm{C} + \delta\mathrm{C},$$
$$\therefore \quad \mathrm{S}\sigma\delta\rho = -\delta\mathrm{C}.$$

From the differential equation, we see that σ is perpendicular to $d\rho$, any tangent vector to the surface; hence σ is a vector normal to the surface.

Since $\delta\rho$, the normal vector between two consecutive surfaces, is in the direction of σ, we have $\nabla\sigma\delta\rho = 0$.

Hence, since
$$\mathrm{S}\sigma\delta\rho = -\delta\mathrm{C},$$
$$\delta\rho = -\sigma^{-1}\delta\mathrm{C},$$

which shows that σ or $\nabla F(\rho)$, besides being normal to the surface, is in length inversely proportional to the normal distance between two consecutive surfaces (see Tait, "Quaternions," p. 215).

The physical interpretation of this is that the lines of flow are perpendicular to the surfaces $F(\rho) = C$,

and the tensor of σ, the vector velocity. must of course vary inversely as the normal distance between two consecutive surfaces.

It is to be observed that $\int S\sigma d\rho$ is what Thomson and Tait (*Nat. Phil.*, p. 125) have called the "displacement-function," or the "entire tangential displacement" along any curve from a fixed point up to P.

In the present case $\int S\sigma d\rho = F(\rho) - F(0)$.

If the curve be closed, we can transform the expression $\int S\sigma d\rho$ into the form (Tait, *Edin. Trans.*, Vol. xxvi., p. 75)

$$\iint d\varsigma S \,.\, \nabla\sigma U\nu,$$

where $d\varsigma$ is the element of any surface bounded by the closed curve, and ν is the normal to $d\varsigma$. In the case of

$$\int S\sigma d\rho = \text{perfect differential},$$

$$\iint d\varsigma S \,.\, \nabla\sigma U\nu = 0,$$

as indeed is otherwise evident, since we have $\nabla \,.\, \nabla\sigma = 0$ at every point of the surface. We might therefore have followed the reverse process, viz., first transformed $\int S \,.\, \sigma d\rho$ to the double integral form, and thus deduced that

$$\int S \,.\, \sigma d\rho = 0,$$

or

$$S \,.\, \sigma d\rho = \text{perfect differential}.$$

Case II.—*Motion Rotational only.*

Perhaps the easiest way to treat this is to revert to Cartesians, and we find that

$$u_x = 0, \qquad u_y = 0, \qquad u_z = 0,$$
$$u_y = -v_x, \quad u_z = -w_x, \quad w_y = -v_z.$$

where the suffixes indicate differentiation with regard to that letter.

From these equations we deduce by differentiation

$$u_{yz} = 0, \quad v_{zx} = 0, \quad w_{xy} = 0.$$

Hence, by the previous relations,

$$\begin{aligned} u &= Ay - Bz + D, \\ v &= Bz - Cx + E, \\ w &= Cx - Ay + F, \end{aligned}$$

or the fluid moves as a rigid body.

3. By means of the preceding results, we may at once deduce Stokes' well-known analysis of fluid motion given in *Camb. Phil. Trans.*, Vol. viii., p. 290 *sqq.*

We observe that what we have called $\sigma' - \sigma$ is the "relative velocity" of a point of the fluid about P; and our analysis has shewn that this relative motion is compounded of a motion of rotation about P given by $V \cdot \epsilon\pi$ where ϵ is known, and a pure strain given by the expression $\bar{\phi}\pi - \pi$. Taking this latter part only, and using known quaternion results, we see that we may write

$$\bar{\phi}\pi - \pi = -g_1 i S i\pi - g_2 j S j\pi - g_3 k S k\pi,$$

where i, j, k are unit-vectors, and are taken as the roots of the equation

$$V \cdot \pi(\bar{\phi}\pi - \pi) = 0,$$

or of

$$V \cdot \pi\bar{\phi}\pi = 0;$$

(see Tait, "Quaternions," p. 91); and further, $\bar{\phi}$ being self-conjugate, i, j, k form in general a real and definite rectangular system. In other words, there are in general three directions at right angles to each other in which the point Q may be taken, so as to leave the line PQ unaltered in direction after the strain $(\bar{\phi}\pi - \pi)$ has taken place. The quantities g_1, g_2, g_3 are inversely proportional to the squares of the axes

$$S\pi(\bar{\phi}\pi - \pi) = 1.$$

We may, with Stokes, call these three directions *the axes of extension.*

This mode of exhibiting the values of ϵ and of the axes of extension shows clearly the independence of these quantities of the axes of coordinates.

We might break up the quantity $\bar{\phi}\pi$ into many different forms; and in fact, in treating special cases of fluid motion, it would be important to select a suitable mode of resolution. Tait gives, amongst other forms, the following

$$\bar{\phi}\pi = \frac{(\iota - \kappa)^2\pi + 2(\iota S\kappa\pi + \kappa S\iota\pi)}{(\kappa^2 - \iota^2)^2},$$

or

$$= f\pi + V \cdot \lambda\pi\mu,$$

or

$$= a\alpha V\alpha\pi + b\beta S\beta\pi,$$

all of which we may interpret easily by remembering that

$c\pi$ denotes a simple contraction or dilatation;

$\beta \mathrm{S}\alpha\pi$ denotes the alteration in π due to a simple shear where $\mathrm{T}\alpha=1$, $\mathrm{T}\beta=$ amount of shear at unit distance, and $\mathrm{S}\alpha\beta=0$.

$mi\mathrm{S}i\pi$ denotes an elongation of all lines parallel to i in the ratio m.

Stokes has chosen (for a reason to be noticed presently) a resolution, which we may write

$$\bar{\varphi}\pi-\pi=-c\pi+m\,(i\mathrm{S}i\pi-j\mathrm{S}j\pi)+n\,(i\mathrm{S}i\pi-k\mathrm{S}k\pi),$$

where c, m, n are scalar constants.

On comparing this expression with that written above, viz.,

$$\bar{\varphi}\pi-\pi=-g_1 i\mathrm{S}i\pi-g_2 j\mathrm{S}j\pi-g_3 k\mathrm{S}k\pi,$$

we find (with him)

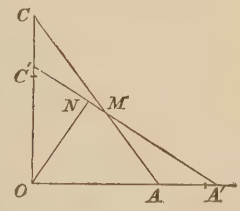
$$-c=\tfrac{1}{3}(g_1+g_2+g_3),\quad -m=\tfrac{1}{3}(g_1+g_3-2g_2),\quad -n=\tfrac{1}{3}(g_1+g_2-2g_3).$$

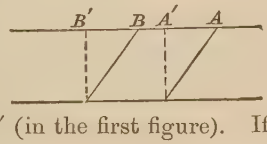
It is easy to see, by Cartesians, what this analysis means. It means that the motion due to $\bar{\varphi}\pi-\pi$ is compounded of a motion of uniform dilatation, and two elementary shearing motions in the plane of (xy), (xz), omitting the rotational parts of the shears.

For a shear in the plane of (xy) may, omitting rotation, be represented by elongating all lines parallel to x in the ratio $1+m$, and contracting all lines parallel to y in the ratio $\dfrac{1}{1+m}$, or $1-m$ if m be small. That is, the alteration in x due to the irrotational part of the shear is mx, and the alteration in y is $-my$; which is precisely what has been written down above.

If we wish to deduce from the expression $\beta \mathrm{S}\alpha\pi$ (the general expression for a shear in the plane of α and β, as above explained) the actual value of the alteration in π, we must remember—

(i.) The shear may be affected by elongating lines in the direction parallel to OA in the ratio $1+m$, and contracting lines in the direction parallel to OC in the ratio $1-m$, followed by a rotation. By means of the elongation and contraction the line CA comes into the position C′A′, where A and C are taken so that OA=OC′, OA′= OC. To bring the body now into the same position in space as it would have been had the shear been effected by a sliding of parallel planes over each other, as indicated in the annexed diagram (see Thomson and Tait, p. 106), we must give the body or element under consideration a rotation as a rigid body through the angle CMC′ (in the first figure). If,





then, we wish to find the irrotational part of $\pi+\beta S\alpha\pi$, we must rotate the body back through the angle CMC′, *i.e.*, we must substitute in the expression $\pi+\beta S\alpha\pi$, for α, $\alpha-V\epsilon\alpha$; for β, $\beta-V\epsilon\beta$; for π, $\pi-V\epsilon\pi$; where ϵ is the vector axis of rotation.

(ii.) β and α, in the expression $\beta S\alpha\pi$, are the lines C′A′ and ON respectively.

Hence we have $CC' = m\mathrm{OA}$,

$$\angle \mathrm{CMC}' = \frac{m\mathrm{OA}}{\frac{1}{2}\mathrm{CA}} \cdot \frac{\mathrm{OA}}{\mathrm{CA}} = \frac{m\mathrm{OA}^2}{\frac{1}{2}\mathrm{CA}^2},$$

but

$$\mathrm{CA}^2 = \mathrm{OA}^2\{1+(1+m)^2\}$$
$$= 2\,\mathrm{OA}^2(1+m),$$
$$\therefore \quad \angle \mathrm{CMC}' = m \text{ to a first approximation.}$$

Now if we consider the plane (ij) as the plane of α and β, we have

$$\alpha = \frac{1}{\sqrt{2}}\{i(1-m)+j(1+m)\} = \frac{i+j}{\sqrt{2}} - m\frac{i-j}{\sqrt{2}},$$
$$\beta = m\sqrt{2}\{i(1+m)-j(1-m)\} = m\sqrt{2}(i-j),$$

since $T\alpha=1$, $T\beta=2m=$ amount of shear at unit distance (see Thomson and Tait, p. 107), $S\alpha\beta=0$.

Also, $\epsilon = mk$, and $V\epsilon\pi = -mVk(iSi\pi+jSj\pi) = -m(jSi\pi-iSj\pi)$.

Thus the expression $\pi+\beta S\alpha\pi$ becomes

$$\pi-V\epsilon\pi+(\beta-V\epsilon\beta)\,S(\alpha-V\epsilon\alpha)(\pi-V\epsilon\pi)$$
$$= \pi+m(jSi\pi-iSj\pi)+m\sqrt{2}(i-j)\,S\frac{i+j}{\sqrt{2}}[\pi+m(jSi\pi-iSj\pi)]$$
$$= \pi+m(jSi\pi-iSj\pi)+m(iSi\pi-jSj\pi)+m(iSj\pi-jSi\pi)$$
$$= \pi+m(iSi\pi-jSj\pi),$$

which gives the alteration in the value of π in the form before found.

4. Let us treat the case of an incompressible fluid when the vector axis of rotation is supposed given at each point of the fluid.

We have to solve the equations

$$V.\nabla\sigma = \alpha, \text{ when } \alpha \text{ is given at each point of the fluid} \ldots\ldots(1),$$
$$S.\nabla\sigma = 0 \ldots\ldots(2).$$

We may assume $\sigma = \nabla A + V.\nabla\mathfrak{A}$,

where A is a scalar function, $\mathfrak{A}$ a vector that we desire to find.

By (2) we have, since $S\nabla V.\nabla\mathfrak{A} = S\nabla^2\mathfrak{A} = 0$ (∇^2 being a scalar operator),

$$\nabla^2 A = 0 \ldots\ldots(3).$$

By substitution for σ in (1),

$$V.\nabla V.\nabla\mathfrak{A} = \alpha,$$

or

$$\nabla^2\mathfrak{A} - \nabla S.\nabla\mathfrak{A} = \alpha.$$

To make the problem workable, we must suppose in addition that

$$S \cdot \nabla \mathfrak{A} = 0,$$

and then

$$\nabla^2 \mathfrak{A} = \alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Equations (3) and (4) can be solved by means of potential properties, and by the nature of the potential admit of only one solution. If then ρ = vector joining the points (xyz) and $(x'y'z')$, we have

$$\iiint \nabla^2 \frac{1}{T\rho}\, dx'dy'dz' = 0$$

if the space over which the integration takes place does not contain (xyz);

and

$$\iiint \nabla^2 \frac{1}{T\rho}\, dx'dy'dz' = -4\pi$$

if the space contains (xyz).

Consequently, if α' be the value of α at the point $(x'y'z')$, and m be a scalar quantity to be determined by the boundary conditions,

$$A = \iiint \frac{m}{T\rho}\, dx'dy'dz',$$

the limits being taken so that the integration extends over all space exclusive of the space considered;

$$\mathfrak{A} = -\frac{1}{4\pi} \iiint \frac{\alpha'}{T\rho}\, dx'dy'dz',$$

the integration being extended over all the space considered.

To make the distribution of m definite, we may impose the condition that m shall be distributed entirely on the surface of the space considered. To show that it is possible for m to be a superficial distribution, we proceed thus. Let ν be the normal to the bounding surface; then the problem is impossible, unless

$$V \cdot a\nu = 0 \quad \text{at the boundary,}$$

or

$$S \cdot \sigma U\nu = 0;$$

or, putting for σ its value

$$\sigma = \nabla A + \nabla \mathfrak{A} \quad (S \cdot \nabla \mathfrak{A} = 0),$$

we have $S \cdot \nabla A U\nu = -S \cdot \nabla \mathfrak{A} U\nu =$ a known quantity.

Let A be the potential of matter distributed on the surface; it is always possible to determine such a distribution that the normal component of the force shall have assigned values all over the surface; that is, m may be a superficial distribution.

We may now show how it is that $S \cdot \nabla \mathfrak{A} = 0$.

Let us put

$$dx'dy'dz' = d\omega',$$

$$\nabla' = i\frac{d}{dx'} + j\frac{d}{dy'} + k\frac{d}{dz'},$$

so that $$\nabla' \frac{1}{\mathrm{T}\rho} = -\nabla \frac{1}{\mathrm{T}\rho};$$

then $$\mathrm{S} \cdot \nabla \mathfrak{A} = -\frac{1}{4\pi} \iiint \mathrm{S} \cdot \nabla \left(\alpha' \frac{1}{\mathrm{T}\rho} \right) d\omega'$$

$$= \frac{1}{4\pi} \iiint \mathrm{S} \cdot \alpha' \nabla' \frac{1}{\mathrm{T}\rho} d\omega'.$$

But, by the Quaternion form of Green's Theorem (Tait, *Edin. Trans.*, Vol. xxvi.), we have, if P, P′ be any two scalar functions,

$$\iiint \mathrm{S} \cdot \nabla' \mathrm{P} \, \nabla' \mathrm{P}' d\omega' = \iint \mathrm{P}' \mathrm{S} \cdot \nabla' \mathrm{P} \mathrm{U} \nu \, d\varsigma' - \iiint \mathrm{P}' \nabla'^2 \mathrm{P} \, d\omega';$$

or, if $\nabla' \mathrm{P} = \alpha'$, $\mathrm{P}' = \frac{1}{\mathrm{T}\rho}$,

$$\iiint \mathrm{S} \cdot \alpha' \nabla' \frac{1}{\mathrm{T}\rho} d\omega' = \iint \frac{1}{\mathrm{T}\rho} \mathrm{S} \cdot \alpha' \mathrm{U} \nu \, d\varsigma' - \iiint \frac{1}{\mathrm{T}\rho} \mathrm{S} \cdot \nabla' \alpha' d\omega'.$$

Evidently $\mathrm{S} \cdot \alpha' \mathrm{U} \nu = 0$ over the boundary,

and $\mathrm{S} \cdot \nabla' \alpha' = \mathrm{S} \cdot \nabla' (\mathrm{V} \cdot \nabla' \sigma' + \mathrm{S} \cdot \nabla' \sigma') = \mathrm{S} \cdot \nabla' (\nabla' \sigma') = 0$ everywhere.

Hence $$\mathrm{S} \cdot \nabla \mathfrak{A} = 0.$$

Consequently we have, finally,

$$\sigma = \nabla \iiint \frac{m}{\mathrm{T}\rho} d\omega' - \frac{1}{4\pi} \nabla \iiint \frac{\alpha'}{\mathrm{T}\rho} d\omega' \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The result we have just arrived at, viz.,

$$\sigma = \nabla (\mathrm{A} + \mathfrak{A}),$$

where A and $\mathfrak{A}$ are found as above, may be usefully interpreted as follows:—Call A the scalar velocity potential, $\mathfrak{A}$ the vector velocity potential; then the velocity at any point of a fluid in motion depends partly on the scalar potential, partly on the vector potential. That part depending on the scalar potential corresponds to the pure strain in our previous analysis, and may be called the irrotational motion. That part due to the vector potential (involving α) corresponds to the motion of rotation, and may be called the vortex motion.

We thus see more clearly the meaning of the term "velocity potential" which is used in the ordinary treatment of fluid motion, viz., the case in which $\mathfrak{A}$ is supposed to vanish, giving $\mathrm{V} \cdot \nabla \sigma = 0$ at every point, or the motion irrotational. It is customary to speak of this being the only case of motion in which a "velocity potential" exists. It would seem more appropriate to say that in this case the vector velocity potential vanishes, and we are left with the scalar velocity potential alone, as in the case I have considered in the earlier part of this paper.

It may be interesting to trace some analogies here; *e. g.* (Maxwell, "Electricity," Vol. ii.), if $\mathfrak{B}$ be the magnetic induction, and $\mathfrak{B} = \nabla\mathfrak{A}$, then $\mathfrak{A}$ is called the vector potential.

We may bring out the nature of the two quantities, which Maxwell there calls the scalar potential and the vector potential, thus:—Let $\mathfrak{I}$ be the vector magnetisation, ρ be the vector joining the points (xyz), $(\xi\eta\zeta)$ (in Maxwell's notation), q the quaternion $\mathfrak{I}\rho^{-1}$; then

$$\text{Scalar potential} = \iiint \frac{\mathrm{S}q}{\mathrm{T}\rho}\,dx\,dy\,dz,$$

$$\text{Vector potential} = \iiint \frac{\mathrm{V}q}{\mathrm{T}\rho}\,dx\,dy\,dz.$$

If here m be a superficial distribution, and such that we may write $m = \mathrm{S}\cdot\nabla\sigma'$ at the surface, while everywhere else $\mathrm{S}\cdot\nabla\sigma' = 0$, then our previous results show us that

$$\text{A (scalar potential)} = \iiint \frac{\mathrm{S}\cdot\nabla\sigma'}{\mathrm{T}\rho}\,dx'dy'dz',$$

$$\mathfrak{A}\text{ (vector potential)} = \iiint \frac{\mathrm{V}\cdot\nabla\sigma'}{\mathrm{T}\rho}\,dx'dy'dz'.$$

Let us evaluate the part of the velocity at (xyz) due to the rotation of the element at $(x'y'z')$. Call it $\delta\sigma$; then, looking at the last term of equation (5), we have

$$\delta\sigma = -\frac{1}{4\pi}\mathrm{V}\cdot\nabla\left(\alpha'\frac{1}{\mathrm{T}\rho}\right)d\omega'$$

[we must here put in the V because it is only when we integrate over the surface that $\mathrm{S}\cdot\nabla\mathfrak{A} = 0$],

or
$$\delta\sigma = -\frac{1}{4\pi}\mathrm{V}\cdot\alpha'\nabla\left(\frac{1}{\mathrm{T}\rho}\right)d\omega';$$

but
$$\nabla\left(\frac{1}{\mathrm{T}\rho}\right) = \frac{\rho}{(\mathrm{T}\rho)^3},$$

$$\therefore\quad \delta\sigma = -\frac{1}{4\pi\,(\mathrm{T}\rho)^3}\mathrm{V}\cdot\alpha'\rho\,d\omega'.$$

Hence
$$\mathrm{S}\cdot\alpha'\delta\sigma = 0,$$

$$\mathrm{S}\cdot\rho\,\delta\sigma = 0;$$

or the direction of the velocity at any point due to the rotation of an element at any other point is perpendicular to the plane containing the first point and the element of the vortex line through the second point.

Also, if θ be the angle between ρ and α',

$$\left.\begin{matrix}\text{magnitude of velocity at } (xyz)\\ \text{due to rotation at } (x'y'z')\end{matrix}\right\} = \frac{\mathrm{T}\alpha'}{4\pi\,(\mathrm{T}\rho)^2}\sin\theta.$$

February 8th, 1877.

C. W. MERRIFIELD, Esq., F.R.S., Vice-President, in the Chair.

Mr. G. W. Von Tunzelmann was admitted into the Society.

Mr. A. B. Kempe moved, and Mr. Roberts seconded, a vote of thanks to Lord Rayleigh and Mr. Spottiswoode for the presents of their portraits to the Society.

The following communications were made:—

"On the Area of the Quadrangle formed by the Four Points of Intersection of Two Conics:" Mr. Leudesdorf (read in part by Mr. Tucker).

"On a Certain Series:" Mr. J. W. L. Glaisher.

"On the Equation $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$:" Prof. Cayley.

"Classification of Loci, and a Theorem in Residuation:" Prof. Clifford.

The following presents were made to the Society:—

"Reprint from Educational Times," Vol. xxvi.

"Educational Times," Feb.

"Nautical Almanac" for 1880, presented by Lords Commissioners of Admiralty.

"Haversines, Natural and Logarithmic, used in computing Lunar Distances for the Nautical Almanac," edited by Major-General Hannyngton, F.R.A.S. 1876.

"Proceedings of the Royal Society," Vol. xxv., No. 177.

"Vorlesungen über Geometrie," von Alfred Clebsch, bearbeitet und herausgegeben von Dr. F. Lindemann, ersten Bandes zweiter Theil; Leipzig, 1876: from the Editor.

"Weitere Untersuchungen über das Ikosaeder II.," von F. Klein, Erlangen (read Jan. 15, 1877).

On the General Differential Equation $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$, *where* X, Y *are the same Quartic Functions of* x, y *respectively.* *By* Prof. Cayley, F.R.S.

[*Read February 8th*, 1877.]

Write $\Theta=a+b\theta+c\theta^2+d\theta^3+e\theta^4$, the general quartic function of θ; and let it be required to integrate by Abel's theorem the differential equation

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0.$$

We have
$$\begin{vmatrix} x^2, & x, & 1, & \sqrt{X} \\ y^2, & y, & 1, & \sqrt{Y} \\ z^2, & z, & 1, & \sqrt{Z} \\ w^2, & w, & 1, & \sqrt{W} \end{vmatrix} = 0,$$

a particular integral of

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}+\frac{dw}{\sqrt{W}}=0;$$

and consequently the above equation, taking therein z, w as constants, is the general integral of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$,

viz., the two constants z, w must enter in such wise that the equation contains only a single constant; whence also, attributing to w any special value, we have the general integral with z as the arbitrary constant.

Take $w=\infty$; the equation becomes

$$\begin{vmatrix} x^2, & x, & 1, & \sqrt{X} \\ y^2, & y, & 1, & \sqrt{Y} \\ z^2, & z, & 1, & \sqrt{Z} \\ 1, & 0, & 0, & \sqrt{e} \end{vmatrix} = 0,$$

a relation between x, y, z which may be otherwise expressed by means of the identity

$$e(\theta^2+\beta\theta+\gamma)^2-(e\theta^4+d\theta^3+c\theta^2+b\theta+a) = (2\beta e-d)(\theta-x)(\theta-y)(\theta-z),$$

or, what is the same thing,

$$\begin{aligned} e(2\gamma+\beta^2)-c &= -(2\beta e-d)(x+y+z), \\ e\ 2\beta\gamma\quad -b &= \quad (2\beta e-d)(yz+zx+xy), \\ e\ \gamma^2\quad\ -a &= -(2\beta e-d)xyz, \end{aligned}$$

where β, γ are indeterminate coefficients which are to be eliminated.

Write $x^2-\frac{\sqrt{X}}{\sqrt{e}}=P$, $y^2-\frac{\sqrt{Y}}{\sqrt{e}}=Q$;

then we have

$$\begin{aligned} &\beta x+\gamma+P=0, \quad \text{giving} \quad \beta:\gamma:1=Q-P:Py-Qx:x-y; \\ &\beta y+\gamma+Q=0; \end{aligned}$$

and substituting these values in the first of the preceding three equations, we have

$$e\frac{2(Py-Qx)(x-y)+(Q-P)^2}{(x-y)^2}-c=-\left(\frac{2(Q-P)e}{x-y}-d\right)(x+y+z),$$

that is,

$$e\left\{\frac{2(Qy-Px)}{x-y}+\frac{(Q-P)^2}{(x-y)^2}+\frac{2(Q-P)}{x-y}z\right\}=c+d(x+y+z);$$

or, reducing by

$$Qy-Px = y^3-x^3+\frac{x\sqrt{X}-y\sqrt{Y}}{\sqrt{e}},$$

$$Q-P = y^2-x^2+\frac{\sqrt{X}-\sqrt{Y}}{\sqrt{e}}, = y^2-x^2+(y-x)\frac{M}{\sqrt{e}}, \text{ if } M=\frac{\sqrt{X}-\sqrt{Y}}{x-y},$$

this is

$$e\left\{\frac{2(x\sqrt{X}-y\sqrt{Y})}{\sqrt{e}(x-y)}+2xy+\frac{M^2}{e}-2(x+y)\frac{M}{\sqrt{e}}-2(x+y)z+2z\frac{M}{\sqrt{e}}\right\}$$
$$= c+d(x+y+z)+e(x+y)^2.$$

We have Euler's solution in the far more simple form

$$M^2 = C+d(x+y)+e(x+y)^2,$$

where C is the arbitrary constant. It is to be observed that, in the particular case where $e=0$, the first equation becomes

$$M^2 = c+d(x+y+z)+e(x+y)^2;$$

and the two results agree on putting $C = c+dz$.

But it is required to identify the two solutions in the general case where e is not $= 0$. I remark that I have, in my Treatise on Elliptic Functions, further developed the theory of Euler's solution, and have shown that, regarding C as variable, and writing

$$\mathfrak{C} = ad^2+b^2e-2bcd+C[-4ae+bd+(C-c)^2],$$

then the given equation between the variables x, y, C corresponds to the differential equation

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dC}{\sqrt{\mathfrak{C}}}=0,$$

a result which will be useful for effecting the identification. The Abelian solution may be written

$$e\left\{\frac{2(x\sqrt{X}-y\sqrt{Y})}{\sqrt{e}(x-y)}-x^2-y^2+\frac{M^2}{e}-2(x+y)\frac{M}{\sqrt{e}}\right\}-c-d(x+y)$$
$$= z\{d+2e(x+y)-2M\sqrt{e}\};$$

and substituting for M its value, and multiplying by $(x-y)^2$, the equation becomes

$$2\sqrt{e}(x-y)(x\sqrt{X}-y\sqrt{Y})-e(x^2+y^2)(x-y)^2+(\sqrt{X}-\sqrt{Y})^2$$
$$-2(x^2-y^2)(\sqrt{X}-\sqrt{Y})\sqrt{e}-c(x-y)^2-d(x+y)(x-y)^2$$
$$= z(x-y)\{d(x+y)+2e(x^2-y^2)-2(\sqrt{X}-\sqrt{Y})\sqrt{e}\}.$$

On the left-hand side the rational part is

$$X+Y+c(-x^2+2xy-y^2)+d(-x^3+x^2y+xy^2-y^3)$$
$$+e(-x^4+2x^3y-2x^2y^2+2xy^3-y^4),$$

which, substituting therein for X, Y their values, becomes

$$= 2a+b(x+y)+c\,.\,2xy+d\,xy\,(x+y)+e\,.\,2xy\,(x^2-xy+y^2);$$

and the irrational part is at once found to be

$$= 2\sqrt{e}\,(x-y)\,(x\sqrt{Y}-y\sqrt{X})-2\sqrt{XY}.$$

The equation thus is

$$z=\frac{\begin{array}{c}2a+b\,(x+y)+c\,.\,2xy+d\,xy\,(x+y)+e\,.\,2xy\,(x^2-xy+y^2)\\ +2\sqrt{e}\,(x-y)\,(x\sqrt{Y}-y\sqrt{X})-2\sqrt{XY}\end{array}}{(x-y)\,\{d\,(x-y)+2e\,(x^2-y^2)-2\,(\sqrt{X}-\sqrt{Y})\,\sqrt{e}\}},$$

which equation is thus a form of the general integral of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$, and also a particular integral of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0$.

Multiplying the numerator and denominator by

$$d\,(x-y)+2e\,(x^2-y^2)+2\,(\sqrt{X}-\sqrt{Y})\,\sqrt{e},$$

the denominator becomes

$$=(x-y)^3\left\{d+2e\,(x+y)^2-4e\left(\frac{\sqrt{X}-\sqrt{Y}}{x-y}\right)^2\right\},$$

which, introducing herein the C of Euler's equation, is

$$=(x-y)^3\,(d^2-4e\mathrm{C}).$$

We have therefore

$$\begin{aligned}z\,(x-y)^3\,(d^2-4e\mathrm{C})\\ =\{&2a+b\,(x+y)+c\,.\,2xy+d\,xy\,(x+y)+e\,.\,2xy\,(x^2-xy+y^2)\\ &+2\sqrt{e}\,(x-y)\,(x\sqrt{Y}-y\sqrt{X})-2\sqrt{XY}\}\\ \times\{&d\,(x-y)+2e\,(x^2-y^2)+2\sqrt{e}\,(\sqrt{X}-\sqrt{Y})\}.\end{aligned}$$

Using $\mathfrak{C}$ to denote the same value as before, the function on the right hand is in fact

$$=(x-y)^3\,\{2be-cd+d\mathrm{C}+2\sqrt{e}\,\sqrt{\mathfrak{C}}\}\,;$$

and, this being so, the required relation between z, C is

$$z\,(d^2-4e\mathrm{C})=\{2be-cd+d\mathrm{C}+2\sqrt{e}\,\sqrt{\mathfrak{C}}\}.$$

To prove this, we have first, from the equation

$$\left(\frac{\sqrt{X}-\sqrt{Y}}{x-y}\right)^2=\mathrm{C}+d\,(x+y)+e\,(x+y)^2,$$

to express $\mathfrak{C}$ as a function of x, y. This equation, regarding therein C as a variable, gives

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{d\mathrm{C}}{\sqrt{\mathfrak{C}}}=0\,;$$

and we have therefore

$$-\sqrt{\mathfrak{C}}=\sqrt{X}\,\frac{d\mathrm{C}}{dx}=\sqrt{Y}\,\frac{d\mathrm{C}}{dy},$$

viz., $\sqrt{X}\frac{dC}{dx}$ will be a symmetrical function of x, y. Putting, as before, $M = \frac{\sqrt{X}-\sqrt{Y}}{x-y}$, we have

$$C = M^2 - d(x+y) - e(x+y)^2,$$

and thence

$$\frac{dC}{dx} = 2M\frac{dM}{dx} - d - 2e(x+y).$$

We have

$$\frac{dM}{dx} = \frac{1}{x-y}\frac{X'}{2\sqrt{X}} - \frac{\sqrt{X}-\sqrt{Y}}{(x-y)^2},$$

and hence

$$\begin{aligned}\sqrt{\mathfrak{C}}(x-y)^3 &= -\sqrt{X}(x-y)^3\left\{2M\frac{dM}{dx} - d - 2e(x+y)\right\}\\ &= -(x-y)X'(\sqrt{X}-\sqrt{Y}) + 2(X+Y-2\sqrt{XY})\sqrt{X}\\ &\qquad + (d+2e\overline{x+y})(x-y)^3\sqrt{X}\\ &= [(x-y)X' + 2X + 2Y + (d+2e\overline{x+y})(x-y)^3]\sqrt{X}\\ &\quad + [(x-y)X' - 4X]\sqrt{Y}.\end{aligned}$$

We obtain at once the coefficient of $\sqrt{Y}$, and with little more difficulty that of $\sqrt{X}$; and the result is

$$\begin{aligned}\sqrt{\mathfrak{C}}(x-y)^3 = &-[4a+3bx+2cx^2+dx^3+y(b+2cx+3dx^2+4ex^3)]\sqrt{Y}\\ &+[4a+3by+2cy^2+dy^3+x(b+2cy+3dy^2+4ey^3)]\sqrt{X}.\end{aligned}$$

We have also

$$\begin{aligned}C(x-y)^2 &= (\sqrt{X}-\sqrt{Y})^2 - d(x+y)(x-y)^2 - e(x+y)^2(x-y)^2\\ &= X+Y-d(x^3-x^2y-xy^2+y^3) - e(x^4-2x^2y^2+y^4) - 2\sqrt{XY}\\ &= 2a+b(x+y)+c(x^2+y^2)+dxy(x+y)+2ex^2y^2-2\sqrt{XY},\end{aligned}$$

or say

$$\begin{aligned}C(x-y)^3 = &\,2a(x-y)+b(x^2-y^2)+c(x^3-x^2y+xy^2-y^3)+dxy(x^2-y^2)\\ &+2ex^2y^2(x-y)-2(x-y)\sqrt{XY},\end{aligned}$$

and we can hence form the expression of

$$(x-y)^3\{2be-cd+dC+2\sqrt{e}\sqrt{\mathfrak{C}}\},$$

viz., this is

$$\begin{aligned}&= (2be-cd)(x-y)^3\\ &\quad +2ad(x-y)+bd(x^2-y^2)+cd(x^3-x^2y+xy^2-y^3)+d^2xy(x^2-y^2)\\ &\qquad +2de x^2y^2(x-y)-2d(x-y)\sqrt{XY}\\ &\quad +2\sqrt{e}\{[-(4a+3bx+2cx^2+dx^3)-y(b+2cx+3dx^2+4ex^3)]\sqrt{Y}\\ &\qquad +[(4a+3by+2cy^2+dy^3)+x(b+2cy+3dy^2+4ey^3)]\sqrt{X}\},\end{aligned}$$

and this should be

$$\begin{aligned}&= \{2a+b(x+y)+c.2xy+dxy(x+y)+e.2xy(x^2-xy+y^2)\\ &\qquad +2\sqrt{e}(x-y)(x\sqrt{Y}-y\sqrt{X})-2\sqrt{XY}\}\\ &\qquad \times\{d(x-y)+2e(x^2-y^2)+2\sqrt{e}(\sqrt{X}-\sqrt{Y})\}.\end{aligned}$$

The function on the right hand is in fact

$$= \{2a+b(x+y)+c.2xy+d\,xy\,(x+y)+e.2xy\,(x^2-xy+y^2)-2\sqrt{XY}\}$$
$$\times\{d\,(x-y)+2e\,(x^2-y^2)\}$$
$$+4e\,(x-y)\,(\sqrt{X}-\sqrt{Y})\,(x\sqrt{Y}-y\sqrt{X})$$
$$+2\sqrt{e}\,(\sqrt{X}-\sqrt{Y})\,\{2a+b(x+y)+c.2xy+d\,xy\,(x+y)$$
$$+e.2xy\,(x^2-xy+y^2)-2\sqrt{XY}\}$$
$$+2\sqrt{e}\,(x-y)\,(x\sqrt{Y}-y\sqrt{X})\,\{d\,(x-y)+2e\,(x^2-y^2)\},$$

viz., this is

$$= \{2a+b(x+y)+c.2xy+d\,xy\,(x+y)+e.2xy\,(x^2-xy+y^2)\}$$
$$\times[d\,(x-y)+2e\,(x^2-y^2)]+4e\,(x-y)\,(-x\mathrm{Y}-y\mathrm{X})$$
$$-2\sqrt{XY}\,[d\,(x-y)+2e\,(x^2-y^2)]+4e\,(x-y)\,(x+y)\,\sqrt{XY}$$
$$+2\sqrt{e}\left\{\begin{array}{l}\sqrt{X}\{2a+b(x+y)+c.2xy+d\,xy\,(x+y)+e.2xy\,(x^2-xy+y^2)\\ \qquad +2\mathrm{Y}-(x-y)\,y\,[d\,(x-y)+2e\,(x^2-y^2)]\}\\ -\sqrt{Y}\{2a+b(x+y)+c.2xy+d\,xy\,(x+y)+e.2xy\,(x^2-xy+y^2)\\ \qquad +2\mathrm{X}-(x-y)\,x\,[d\,(x-y)+2e\,(x^2-y^2)]\}\end{array}\right\}$$

which is, in fact, equal to the expression on the left-hand side.

To complete the theory, we require to express $\sqrt{Z}$ as a function of x, y. It would be impracticable to effect this by direct substitution of the foregoing value of z; but, observing that the value in question is a solution of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0$, or, what is the same thing, that $\frac{1}{\sqrt{X}}+\frac{1}{\sqrt{Z}}\frac{dz}{dx}=0$, $\frac{1}{\sqrt{Y}}+\frac{1}{\sqrt{Z}}\frac{dz}{dy}=0$, we can from either of these equations, considering therein z as a given function of x, y, calculate $\sqrt{Z}$.

Writing for shortness

$$z=\frac{\mathrm{J}-2\sqrt{e}\,y\,(x-y)\,\sqrt{X}+2\sqrt{e}\,x\,(x-y)\,\sqrt{Y}-2\sqrt{XY}}{\mathrm{R}-2\sqrt{e}\,(x-y)\,\sqrt{X}+2\sqrt{e}\,(x-y)\,\sqrt{Y}},$$

where $$\mathrm{R}=(x-y)^2\,\{d+2e(x+y)\},$$

$$\mathrm{J}=2a+b(x+y)+2c\,xy+d\,xy\,(x+y)+2e\,xy\,(x^2-xy+y^2);$$

or, if for a moment $z=\frac{\mathrm{N}}{\mathrm{D}}$, then

$$\frac{dz}{dx}=\frac{1}{\mathrm{D}^2}\left(\mathrm{D}\frac{d\mathrm{N}}{dx}-\mathrm{N}\frac{d\mathrm{D}}{dx}\right)=-\frac{\sqrt{Z}}{\sqrt{X}},$$

that is, $$\sqrt{Z}=\frac{\sqrt{X}}{\mathrm{D}^2}\left(\mathrm{N}\frac{d\mathrm{D}}{dx}-\mathrm{D}\frac{d\mathrm{N}}{dx}\right),\ =\frac{\Omega}{\mathrm{D}^2}\ \text{suppose};$$

or writing for shortness $X', \mathrm{R}', \mathrm{J}$ to denote the derived functions

$\frac{dX}{dx}$, $\frac{dR}{dx}$, $\frac{dJ}{dx}$ (Y′ is afterwards written to denote $\frac{dY}{dy}$, but as the final formulæ contain only X′, $= \frac{X}{dx}$, and Y′, $= \frac{dY}{dy}$, this does not occasion any defect of symmetry), we find

$$\begin{aligned}\Omega = \quad & N\{R'\sqrt{X} - 2\sqrt{e}\,X - \sqrt{e}\,(x-y)\,X' + 2\sqrt{e}\,\sqrt{XY}\} \\ & -D\{J'\sqrt{X} - 2\sqrt{e}\,yX - \sqrt{e}\,(x-y)\,yX' \\ & \qquad + 2\sqrt{e}\,(2x-y)\,\sqrt{XY} - X'\sqrt{Y}\}\,;\end{aligned}$$

and substituting herein for N, D their values, and arranging the terms, we find $\Omega = \sqrt{e}\,\mathfrak{A} + \mathfrak{B}\sqrt{X} + \mathfrak{C}\sqrt{Y} + \sqrt{e}\,\mathfrak{D}\sqrt{XY}$,

where

$$\begin{aligned}\mathfrak{A} = & -J\{2X + (x-y)\,X'\} \\ & -2(x-y)\,yR'X \\ & -4XY \\ & +Ry\{2X + (x-y)\,X'\} \\ & +2(x-y)\,XJ' \\ & +2(x-y)\,X'Y,\end{aligned}$$

$$\begin{aligned}\mathfrak{B} = & \quad JR' \\ & +2e(x-y)\,y\{2X + (x-y)\,X'\} \\ & +4ex\,(x-y)\,Y \\ & -RJ' \\ & -2e(x-y)\,y\{2X + (x-y)\,X'\} \\ & -4e(x-y)\,(2x-y)\,Y,\end{aligned}$$

$$\begin{aligned}\mathfrak{C} = & -4ey\,(x-y)\,X \\ & -2e(x-y)\,x\{2X + (x-y)\,X'\} \\ & -2R'X \\ & +RX' \\ & +2e(x-y)\,y\{2X + (x-y)\,X'\} \\ & +4e(x-y)\,(2x-y)\,X,\end{aligned}$$

$$\begin{aligned}\mathfrak{D} = & \quad 2J \\ & +2(x-y)\,xR' \\ & +2\{2X + (x-y)\,X'\} \\ & -2(2x-y)\,R \\ & -2(x-y)\,X' \\ & -2(x-y)\,J',\end{aligned}$$

where the terms have been written down as they immediately present themselves, but collecting and arranging, we have

$$\begin{aligned}\mathfrak{A} = & \quad 2X(-J + Ry - 2Y) + (x-y)\{2XJ' + 2X'Y - X'J - 2yR'X + yRX'\} \\ \mathfrak{B} = & \quad JR' - J'R - 4e\,(x-y)^2\,Y, \\ \mathfrak{C} = & -2XR' + X'R + 4e\,(x-y)^2\,X - 2e\,(x-y)^3\,X', \\ \mathfrak{D} = & \quad 2J + 4X - 2Rx + 2(x-y)\,(xR' - R - J').\end{aligned}$$

For reducing these expressions, writing

$$\begin{aligned}M &= d + 2e\,(x+y), \\ \Lambda &= c + d\,(x+y) + e\,(x^2+y^2),\end{aligned}$$

we have $R = (x-y)^2\,M$, and therefore $R' = 2\,(x-y)\,M + 2e\,(x-y)^2$;

also $$J = X + Y - (x-y)^2\,\Lambda\,;$$

also, from the original form,

$$J' = b + 2cy + d\,(2xy + y^2) + e\,(6x^2y - 4xy^2 + 2y^3).$$

The final values are

$$\begin{aligned}
\mathfrak{A} &= -X^2-6XY-Y^2+(x-y)^4\{\Lambda^2+(-b+dxy)\,M+xyM^2\},\\
\mathfrak{B} &= (x-y)M\{4Y+(x-y)Y'\}+2e(x-y)^3Y',\\
\mathfrak{C} &= -(x-y)M\{4X-(x-y)X'\}-2e(x-y)^3X',\\
\mathfrak{D} &= 4(X+Y)+4e(x-y)^4,
\end{aligned}$$

which, once obtained, may be verified without difficulty.

Verification of $\mathfrak{A}$.—The equation is

$$\begin{aligned}
-X^2-6XY-Y^2+(x-y)^4\{\Lambda^2+(-b+dxy)\,M+xyM^2\}&\\
= 2X(-J+Ry-2Y)&\\
+(x-y)\{2XJ'+2X'Y-X'J-2yR'X+yRX'\};&
\end{aligned}$$

or, putting for shortness $\Lambda^2+(-b+dxy)\,M+xyM^2=\nabla$, this is

$$\begin{aligned}
(x-y)^4\nabla = {}& X^2+6XY+Y^2\\
&+2X\{-X-3Y+(x-y)^2\Lambda+(x-y)^2yM\}\\
&+(x-y)\{2XJ'+2X'Y-X'J-2yR'X+yRX'\},\\
= {}& -X^2+Y^2+2(x-y)^2X\Lambda+2(x-y)^2yXM\\
&+(x-y)\{2XJ+2X'Y-X'J-2yR'X+yRX'\};
\end{aligned}$$

we have $-X^2+Y^2=-(X-Y)(X+Y)$, where $X-Y$ divides by $x-y$, $=(x-y)\Omega$ suppose; hence, throwing out $(x-y)$, the equation becomes

$$\begin{aligned}
(x-y)^3\nabla = {}& -\Omega(X+Y)+2(x-y)X\Lambda+2(x-y)yXM\\
&+2XJ'+2X'Y-X'\{X+Y-(x-y)^2\Lambda\}\\
&-2yX\{2(x-y)M+2(x-y)^2e\}+(x-y)^2yMX',\\
= {}& -\Omega(X+Y)+2XJ'-X'(X-Y)\\
&+2(x-y)X\Lambda-2(x-y)yXM\\
&+(x-y)^2X'\Lambda-4(x-y^2)eyX+(x-y)^2yMX'.
\end{aligned}$$

We have $2XJ'=J'(X+Y)+J'(X-Y)$, and hence the first line is $=(-\Omega+J')(X+Y)+J'(X-Y)$; $-\Omega+J'$, as will be shown, divides by $x-y$, or say it is $=(x-y)\Phi$, and, as before, $X-Y$ is $=(x-y)\Omega$; hence, throwing out $(x-y)$, the equation becomes

$$\begin{aligned}
(x-y)^2\nabla = {}& \Phi(X+Y)+\Omega(J'-X')+2X\Lambda-2yXM\\
&+(x-y)\{X'\Lambda-4eyX+yMX'\}.
\end{aligned}$$

We have

$$\Omega=b+c(x+y)+d(x^2+xy+y^2)+e(x^3+x^2y+xy^2+y^3),$$

and thence

$$-\Omega+J'=c(-x+y)+d(-x^2+xy)+e(-x^3+5x^2y-5xy^2+y^3);$$

or, dividing this by $(x-y)$, we find

$$\Phi = -c-dx-e\,(x^2-4xy+y^2),$$

or, as this may be written, $\Phi = -\Lambda+dy+4exy$. We find, moreover,

$$J'-X' = 2c(-x+y)+d(-3x^2+2xy+y^2)+e(-4x^3+6x^2y-4xy^2+2y^3),$$

which divides by $(x-y)$, the quotient being

$$-2c-d\,(3x+y)-e\,(4x^2-2xy+2y^2),$$

viz., this is $= -2\Lambda-(x-y)\,(d+2ex)$.

Hence the equation is

$$\begin{aligned}(x-y)^2\,\nabla = {} & (X+Y)\,\{-\Lambda+dy+4exy\}+2X\Lambda-2yXM \\ & +(x-y)\,\Omega\,\{-2\Lambda-(x-y)\,(d+2ex)\} \\ & +(x-y)\quad\{\quad X'\Lambda-4eyX+yMX'\quad\}.\end{aligned}$$

The first line is $(X+Y)\,\{-\Lambda+yM+2\,(x-y)\,ye\}+2X\Lambda-2yXM$, which is $=(\Lambda-yM)\,(X-Y)+2\,(x-y)\,ey\,(X+Y)$; hence, throwing out $(x-y)$, the equation becomes

$$\begin{aligned}(x-y)\,\nabla = {} & (\Lambda-yM)\,\Omega+2ey\,(X+Y)-2\Lambda\Omega+X'\Lambda-4eyX+yMX' \\ & -(x-y)\,\Omega\,(d+2ex) \\ = {} & (\Lambda+yM)\,(-\Omega+X')-2ey\,(X-Y)-(x-y)\,\Omega\,(d+2ex).\end{aligned}$$

We have $-\Omega+X' = c(x-y)+d(2x^2-xy-y^2)+e(3x^3-x^2y-xy^2-y^3)$, which is $=(x-y)\,(\Lambda+xM)$; also $(X-Y)=(x-y)\,\Omega$, as before, whence, throwing out $x-y$, the equation is

$$\nabla = (\Lambda+xM)\,(\Lambda+yM)-2ey\Omega-(d+2ex)\,\Omega,$$

that is,

$$\nabla = (\Lambda+xM)\,(\Lambda+yM)-M\Omega;$$

viz., substituting for ∇ its value, reducing, and throwing out the factor M, the equation becomes

$$-b+dxy = (x+y)\,\Lambda-\Omega,$$

which is right.

Verification of $\mathfrak{B}$.—The equation is

$$\begin{aligned}J\,\{2\,(x-y)\,M+2e\,(x-y)^2\}-J'\,(x-y)^2\,M-4e\,(x-y)^2\,Y \\ = 4\,(x-y)\,MY+(x-y)^2\,MY'+2e\,(x-y)^3\,Y',\end{aligned}$$

which, throwing out $(x-y)$, is

$$\begin{aligned}0 = {} & 2M\,(-J+2Y)+(x-y)\,M\,(J'+Y') \\ & +2e\,(x-y)\,(-J+2Y)+2e\,(x-y)^2\,Y'.\end{aligned}$$

Here $-J+2Y$, $=-(X-Y)+(x-y)^2\Lambda$ divides by $(x-y)$: hence, throwing out the factor $x-y$, the equation is

$$0 = \mathrm{M}\{-2b-2c(x+y)-2d(x^2+xy+y^2)-2e(x^3+x^2y+xy^2+y^3)\} + \mathrm{M}(\mathrm{J}'+\mathrm{Y}')+2\mathrm{M}(x-y)\Lambda+2e(-\mathrm{J}+2\mathrm{Y})+2e(x-y)\mathrm{Y}'.$$

In the first and second terms the factor which multiplies M is

$$c(-2x+2y)+d(-2x^2+2y^2)+e(-2x^3+4x^2y-6xy^2+4y^3),$$

which divides by $x-y$; also $-\mathrm{J}+2\mathrm{Y}, = -(\mathrm{X}-\mathrm{Y})+(x-y)^2\Lambda$, divides by $(x-y)$: hence, throwing this factor out, the equation is

$$0 = \mathrm{M}\{-2c+d(-2x-2y)+e(-2x^2+2xy-4y^2)\}+2\mathrm{M}\Lambda + 2e\{-b-c(x+y)-d(x^2+xy+y^2)-e(x^3+x^2y+xy^2+y^3)\} + 2e(x-y)\Lambda+2e\mathrm{Y}'.$$

Here in the top line the coefficient of M is $= e(2xy-2y^2)$: hence, throwing out the constant factor $2e$, the equation is

$$0 = -b-c(x+y)-d(x^2+xy+y^2)-e(x^3+x^2y+xy^2+y^3)+\mathrm{Y}' + (x-y)y\mathrm{M}+(x-y)\Lambda.$$

The top line is

$$= c(-x+y)+d(-x^2-xy+2y^2)+e(-x^3-x^2y-xy^2+3y^3),$$

which divides by $x-y$; throwing out this factor, the equation is

$$0 = -c-d(x+2y)-e(x^2+2xy+3y^2)+\Lambda+y\mathrm{M},$$

which is right.

Verification of $\mathfrak{C}$.—We have

$$-2\mathrm{X}\{2(x-y)\mathrm{M}+2e(x-y)^2\} + (x-y)^2\mathrm{X}'\mathrm{M}+4e(x-y)^2\mathrm{X}-2e(x-y)^3\mathrm{X}' = -(x-y)\mathrm{M}\{4\mathrm{X}-(x-y)\mathrm{X}'\}-2e(x-y)^3\mathrm{X}',$$

which is in fact an identity.

Verification of $\mathfrak{D}$.—The equation may be written

$$4\mathrm{X}+4\mathrm{Y}+4e(x-y)^4 = 2\mathrm{X}+2\mathrm{Y}-2(x-y)^2\Lambda + 4\mathrm{X}-2x(x-y)^2\mathrm{M} + 2(x-y)\{2(x-y)x\mathrm{M}+2ex(x-y)^2-\mathrm{M}(x-y)^2-\mathrm{J}'\},$$

viz., this is

$$0 = 2\mathrm{X}-2\mathrm{Y}-4e(x-y)^4-2(x-y)^2\Lambda+2x(x-y)^2\mathrm{M} + 4ex(x-y)^3-2\mathrm{M}(x-y)^3-2(x-y)\mathrm{J}'.$$

The first term $2(\mathrm{X}-\mathrm{Y})$ divides by $2(x-y)$, or, throwing this factor out, the equation becomes

$$0 = b+c(x+y)+d(x^2+xy+y^2)+e(x^3+x^2y+xy^2+y^3)-\mathrm{J}' -2e(x-y)^3-(x-y)\Lambda+x(x-y)\mathrm{M}+2ex(x-y)^2-\mathrm{M}(x-y)^2.$$

Substituting for J′ its value, the first line becomes

$$c(x-y)+d(x^2-xy)+e(x^3-5x^2y+5xy^2-y^3),$$

which divides by $(x-y)$; hence, throwing out this factor, the equation is

$$0 = c+dx+e(x^2-4xy+y^2)-\Lambda+x\mathrm{M} \\ -2e(x-y)^2+2ex(x-y)-\mathrm{M}(x-y),$$

where the top line is $= d(x-y)+e(2x^2-2xy)$: hence, again throwing out the factor $x-y$, the equation becomes

$$0 = d+2ex-2e(x-y)+2ex-\mathrm{M},$$

which is right.

Recapitulating, we have for the general integral of $\frac{dx}{\sqrt{\mathrm{X}}}+\frac{dy}{\sqrt{\mathrm{Y}}}=0$,

or for a particular integral of $\frac{dx}{\sqrt{\mathrm{X}}}+\frac{dy}{\sqrt{\mathrm{Y}}}+\frac{dz}{\sqrt{\mathrm{Z}}}=0$,

$$z = \frac{\mathrm{J}-2\sqrt{e(x-y)}\,y\sqrt{\mathrm{X}}+2\sqrt{e(x-y)}\,x\sqrt{\mathrm{Y}}-2\sqrt{\mathrm{XY}}}{(x-y)^2\mathrm{M}-2\sqrt{e(x-y)}\sqrt{\mathrm{X}}+2\sqrt{e(x-y)}\sqrt{\mathrm{Y}}},$$

the corresponding value of $\sqrt{\mathrm{Z}}$ being

$$\sqrt{\mathrm{Z}} = \frac{\begin{array}{l}\sqrt{e}\,[-\mathrm{X}^2-6\mathrm{XY}-\mathrm{Y}^2+(x-y)^4\{\Lambda^2+(-b+dxy)\,\mathrm{M}+xy\mathrm{M}^2\}] \\ \quad+[\{4\mathrm{Y}+(x-y)\,\mathrm{Y}'\}\,\mathrm{M}+2e(x-y)^2\mathrm{Y}'](x-y)\sqrt{\mathrm{X}} \\ \quad-[\{4\mathrm{X}-(x-y)\,\mathrm{X}'\}\,\mathrm{M}+2e(x-y)^2\mathrm{X}'](x-y)\sqrt{\mathrm{Y}} \\ \quad+[\qquad 4(\mathrm{X}+\mathrm{Y})+4e(x-y)^4]\qquad\sqrt{\mathrm{XY}}\end{array}}{\{(x-y)^2\mathrm{M}-2\sqrt{e(x-y)}\sqrt{\mathrm{X}}+2\sqrt{e(x-y)}\sqrt{\mathrm{Y}}\}^2},$$

where, as before,

$$\mathrm{M} = d+2e(x+y),$$
$$\Lambda = c+d(x+y)+e(x^2+y^2),$$
$$\mathrm{J} = 2a+b(x+y)+2cxy+dxy(x+y)+exy(x^2-xy+y^2).$$

X is the general quartic function $a+bx+cx^2+dx^3+ex^4$, and Y, Z are the same functions of y, z respectively.

In connexion with what precedes, I give some investigations relating to the more simple form $\Theta = a+c\theta^2+e\theta^4$, or, as it will be convenient to write it, $\Theta = 1-l\theta^2+\theta^4$.

We have

$$\left|\begin{matrix} x, & \sqrt{\mathrm{X}} \\ y, & \sqrt{\mathrm{Y}}\end{matrix}\right| = 0, \text{ a particular integral}$$

$$\left|\begin{matrix} x^3, & x, & \sqrt{\mathrm{X}} \\ y^3, & y, & \sqrt{\mathrm{Y}} \\ z^3, & z, & \sqrt{\mathrm{Z}}\end{matrix}\right| = 0 \text{ the general integral}, \quad \text{a particular integral}$$

$$\left|\begin{matrix} x^3, & x, & x^2\sqrt{\mathrm{X}}, & \sqrt{\mathrm{X}} \\ y^3, & y, & y^2\sqrt{\mathrm{Y}}, & \sqrt{\mathrm{Y}} \\ z^3, & z, & z^2\sqrt{\mathrm{Z}}, & \sqrt{\mathrm{Z}} \\ w^3, & w, & w^2\sqrt{\mathrm{W}}, & \sqrt{\mathrm{W}}\end{matrix}\right| = 0 \text{ the general integral}, \quad \text{a particular integral}$$

$$\cdots \quad \cdots$$

$$\left.\begin{matrix}\left|\begin{matrix} x, & \sqrt{\mathrm{X}} \\ y, & \sqrt{\mathrm{Y}}\end{matrix}\right| = 0, \text{ a particular integral} \\ \left|\begin{matrix} x^3, & x, & \sqrt{\mathrm{X}} \\ y^3, & y, & \sqrt{\mathrm{Y}} \\ z^3, & z, & \sqrt{\mathrm{Z}}\end{matrix}\right| = 0 \text{ the general integral}, \quad \text{a particular integral}\end{matrix}\right\} \text{of } \frac{dx}{\sqrt{\mathrm{X}}}+\frac{dy}{\sqrt{\mathrm{Y}}}=0,$$

$$\left.\begin{matrix}\left|\begin{matrix} x^3, & x, & \sqrt{\mathrm{X}} \\ y^3, & y, & \sqrt{\mathrm{Y}} \\ z^3, & z, & \sqrt{\mathrm{Z}}\end{matrix}\right| = 0 \text{ the general integral}, \quad \text{a particular integral} \\ \left|\begin{matrix} x^3, & x, & x^2\sqrt{\mathrm{X}}, & \sqrt{\mathrm{X}} \\ y^3, & y, & y^2\sqrt{\mathrm{Y}}, & \sqrt{\mathrm{Y}} \\ z^3, & z, & z^2\sqrt{\mathrm{Z}}, & \sqrt{\mathrm{Z}} \\ w^3, & w, & w^2\sqrt{\mathrm{W}}, & \sqrt{\mathrm{W}}\end{matrix}\right| = 0 \text{ the general integral}, \quad \text{a particular integral}\end{matrix}\right\} \text{of } \frac{dx}{\sqrt{\mathrm{X}}}+\frac{dy}{\sqrt{\mathrm{Y}}}+\frac{dz}{\sqrt{\mathrm{Z}}}+\frac{dw}{\sqrt{\mathrm{W}}}=0$$

and so on; viz., in taking

$$\begin{vmatrix} x^3, & x, & \sqrt{X} \\ y^3, & y, & \sqrt{Y} \\ z^3, & z, & \sqrt{Z} \end{vmatrix} = 0 \text{ as the general integral of } \frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0,$$

we consider z as the constant of integration: and so in other cases.

It is to be remarked that it is an essentially different problem to verify a particular integral and to verify a general integral, and that the former is the more difficult one. In fact, if $U=0$ is a particular integral of the differential equation $Mdx+Ndy=0$, then we must have $N\frac{dU}{dx}-M\frac{dU}{dy}=0$, not identically but in virtue of the relation $U=0$, or we have to consider whether two given relations between x and y are in fact one and the same relation. In the case of a general solution, this is theoretically reducible to the form $c=U$, c being the constant of integration, and we have then the equation $N\frac{dU}{dx}-M\frac{dU}{dy}=0$, satisfied identically, or, what is the same thing, U a solution of this partial differential equation.

Hence it is theoretically easier to verify that

$$\begin{vmatrix} x^3, & x, & \sqrt{X} \\ y^3, & y, & \sqrt{Y} \\ z^3, & z, & \sqrt{Z} \end{vmatrix} = 0$$

is a general solution, than to verify that

$$\begin{vmatrix} x, & \sqrt{X} \\ y, & \sqrt{Y} \end{vmatrix} = 0$$

is a particular solution of the differential equation $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$. Moreover, writing the first equation in the before mentioned form

$$-z=\frac{x^2-y^2}{x\sqrt{Y}-y\sqrt{X}},$$

and writing therein $z=\infty$, we see that the second equation $\begin{vmatrix} x, & \sqrt{X} \\ y, & \sqrt{Y} \end{vmatrix} = 0$ is in fact a particular case of the first equation, so that we only require to verify the first equation; or, what is the same thing,

to verify that $$-z=\frac{x^2-y^2}{x\sqrt{Y}-y\sqrt{X}}$$

is the general integral of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}$.

To verify this we have to show that $dz=\Omega\left(\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}\right)$, viz., that

$\sqrt{\mathrm{X}}\dfrac{dz}{dx} = \Omega$, a symmetrical function of (x, y); for then $\sqrt{\mathrm{Y}}\dfrac{dz}{dy} = \Omega$, and we have the relation in question.

We have

$$(x\sqrt{\mathrm{Y}} - y\sqrt{\mathrm{X}})^2 \sqrt{\mathrm{X}}\frac{dz}{dx}$$

$$= \sqrt{\mathrm{X}}\left\{(x^2-y^2)\left(\sqrt{\mathrm{Y}} - \frac{y\mathrm{X}'}{2\sqrt{\mathrm{X}}}\right) - 2x\,(x\sqrt{\mathrm{Y}} - y\sqrt{\mathrm{X}})\right\}$$

$$= \sqrt{\mathrm{X}}\left\{(x^2-y^2-2x^2)\sqrt{\mathrm{Y}} - \frac{(x^2-y^2)\,y\mathrm{X}'}{2\sqrt{\mathrm{X}}} + 2xy\sqrt{\mathrm{X}}\right\}$$

$$= -(x^2+y^2)\sqrt{\mathrm{XY}} + 2xy\mathrm{X} - \tfrac{1}{2}(x^2-y^2)\,y\mathrm{X}'.$$

Writing here $\mathrm{X} = 1 - lx^2 + x^4$, then $\mathrm{X}' = -2lx + 4x^3$, and we have

$$2xy\,(1 - lx^2 + x^4) + (x^2-y^2)\,xy\,(l - 2x^2),$$

$$= xy\,\{2 - 2lx^2 + 2x^4 + (x^2-y^2)(l-2x^2)\},$$

$$= xy\,\{2 - l\,(x^2+y^2) + 2x^2y^2\}.$$

Hence the equation is

$$(x\sqrt{\mathrm{Y}} - y\sqrt{\mathrm{X}})^2\sqrt{\mathrm{X}}\frac{dz}{dx} = -(x^2+y^2)\sqrt{\mathrm{XY}} + xy\,\{2 - l\,(x^2+y^2) + 2x^2y^2\},$$

or we have

$$\Omega = \frac{1}{(x\sqrt{\mathrm{Y}} - y\sqrt{\mathrm{X}})^2}\{-(x^2+y^2)\sqrt{\mathrm{XY}} + xy\,(2 - l\,(x^2+y^2) + 2x^2y^2)\},$$

which is symmetrical in (x, y), as it should be. And observe, further, that since the equation is a particular solution of $\dfrac{dx}{\sqrt{\mathrm{X}}} + \dfrac{dy}{\sqrt{\mathrm{Y}}} + \dfrac{dz}{\sqrt{\mathrm{Z}}} = 0$, we must have $\Omega = -\sqrt{\mathrm{Z}}$; viz., we have

$$\sqrt{\mathrm{Z}}\,(x\sqrt{\mathrm{Y}} - y\sqrt{\mathrm{X}})^2 = -(x^2+y^2)\sqrt{\mathrm{XY}} + xy\,\{2 - l\,(x^2+y^2) + 2x^2y^2\}.$$

Proceeding to the next case, where we have between x, y, z, w a relation which may be written

$$(x^3,\ x,\ x^2\sqrt{\mathrm{X}},\ \sqrt{\mathrm{X}}) = 0,$$

then here a, b, c, d can be determined so that

$$(c\theta^2 + d)^2(1 + \beta\theta^2 + \gamma\theta^4) - (a\theta^3 + b\theta)^2$$

$$= c^2\gamma\,(\theta^2 - x^2)\,(\theta^2 - y^2)\,(\theta^2 - z^2)\,(\theta^2 - w^2),$$

viz., we have $d^2 = c^2\gamma\,x^2y^2z^2w^2$, or say $d = c\sqrt{\gamma}\,xyzw$.

And, supposing the ratios of a, b, c, d determined by the three equations which contain (x, y, z) respectively, we have

$$a : b : c : d$$

$$= (x,\ x^2\sqrt{\mathrm{X}},\ \sqrt{\mathrm{X}}) : -(x^3,\ x^2\sqrt{\mathrm{X}},\ \sqrt{\mathrm{X}}) : (x^3,\ x,\ \sqrt{\mathrm{X}}) : -(x^3,\ x,\ x^2\sqrt{\mathrm{X}}),$$

or in particular

$$\frac{d}{c}=\frac{-(x^3,\ x,\ x^2\sqrt{X})}{(x^3,\ x,\ \sqrt{X})},\qquad =\frac{-xyz\,.\,(x^2,\ 1.\ x\sqrt{X})}{(x^3,\ x,\ \sqrt{X})};$$

whence we have $$w=-\frac{(x^2,\ 1,\ x\sqrt{X})}{(x^3,\ x,\ \sqrt{X})}$$

as a new form of the integral equation; viz., written at full length, this is

$$-w=\begin{vmatrix} x^2, & 1, & x\sqrt{X} \\ y^2, & 1, & y\sqrt{Y} \\ z^2, & 1, & z\sqrt{Z} \end{vmatrix} \div \begin{vmatrix} x^3, & x, & \sqrt{X} \\ y^3, & y, & \sqrt{Y} \\ z^3, & z, & \sqrt{Z} \end{vmatrix};$$

and taking $w=0$ and $=\infty$ respectively, we thus see how

$$\begin{vmatrix} x^2, & 1, & x\sqrt{X} \\ y^2, & 1, & y\sqrt{Y} \\ z^2, & 1, & z\sqrt{Z} \end{vmatrix}=0,\qquad \begin{vmatrix} x^3, & x, & \sqrt{X} \\ y^3, & y, & \sqrt{Y} \\ z^3, & z, & \sqrt{Z} \end{vmatrix}=0,$$

are each of them a particular integral of $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0$.

Reverting to the general form

$$w=-\frac{(x^2,\ 1,\ x\sqrt{X})}{(x^3,\ x,\ \sqrt{X})},$$

this will be a general integral if only

$$dw=\Omega\left(\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}\right),$$

viz., if we have

$$-\sqrt{X}\,\frac{d}{dx}\,\frac{(x^2,\ 1,\ x\sqrt{X})}{(x^3,\ x,\ \sqrt{X})}=\Omega,\ \text{a symmetrical function of } (x, y, z).$$

The expression is

$$\Omega=\frac{1}{(x^3,\ x,\ \sqrt{X})^2}\left\{\{x^2,\ 1,\ x\sqrt{X})\sqrt{X}\,\frac{d}{dx}(x^3,\ x,\ \sqrt{X})\right.$$
$$\left.-(x^3,\ x,\ \sqrt{X})\ \sqrt{X}\,\frac{d}{dx}(x^2,\ 1,\ x\sqrt{X})\right\},$$

or, writing for shortness

$$\begin{aligned} \alpha &= x\,(y^2-z^2), & a &= yz\,(y^2-z^2), \\ \beta &= y\,(z^2-x^2), & b &= zx\,(z^2-x^2), \\ \gamma &= z\,(x^2-y^2), & c &= xy\,(x^2-y^2), \end{aligned}$$

we have

$$\begin{aligned} (x^2,\ 1,\ x\sqrt{X}) &= \alpha\sqrt{X}+\beta\sqrt{Y}+\gamma\sqrt{Z}, \\ (x^3,\ x,\ \sqrt{X}) &= a\sqrt{X}+b\sqrt{Y}+c\sqrt{Z}; \end{aligned}$$

and the formula is

$$(x^3, x, \sqrt{X})^2\Omega$$

$$= (a\sqrt{X}+\beta\sqrt{Y}+\gamma\sqrt{Z})\{(y^3z-yz^3)\tfrac{1}{2}X' + (-3x^2z+z^3)\sqrt{XY}+(3x^2y-y^3)\sqrt{XZ}\}$$
$$-(a\sqrt{X}+b\sqrt{Y}+c\sqrt{Z})\{(y^2-z^2)(X+\tfrac{1}{2}X'x) -2xy\sqrt{XY}-2xz\sqrt{XZ}\}$$

$$= (a\sqrt{X}+\beta\sqrt{Y}+\gamma\sqrt{Z})(L+M\sqrt{XY}+N\sqrt{XZ})$$
$$-(a\sqrt{X}+b\sqrt{Y}+c\sqrt{Z})(P+Q\sqrt{XY}+R\sqrt{XZ}), \text{ suppose,}$$

$$= \frac{\begin{array}{llll} \sqrt{X} & +\sqrt{Y} & +\sqrt{Z} & +\sqrt{XYZ} \end{array}}{\begin{array}{llll} aL & +aMX & +aNX & \\ +\beta MY & +\beta L & & +\beta N \\ +\gamma NZ & & +\gamma L & +\gamma M \\ -aP & -aQX & -aRX & \\ -bQY & -bP & & -bR \\ -cRZ & & -cP & -cQ \end{array}};$$

viz., this is

$$= \{aL-aP+Y(\beta M-bQ)+Z(\gamma N-cR)\}\sqrt{X}$$
$$+\{X(aM-aQ)+\beta L-bP\}\sqrt{Y}$$
$$+\{X(aN-aR)+\gamma L-cP\}\sqrt{Z}$$
$$+(\beta N+\gamma M-bR-cQ)\sqrt{XYZ}.$$

The coefficient of $\sqrt{XYZ}$ is here

$$= y(z^2-x^2)(3x^2y-y^3) \quad , \quad = y^2(z^2-x^2)(3x^2-y^2)$$
$$+z(x^2-y^2)(-3x^2z+z^3) \qquad +z^2(x^2-y^2)(-3x^2+z^2)$$
$$-zx(z^2-x^2)(2xz) \qquad -2x^2z^2(z^2-x^2)$$
$$-xy(x^2-y^2)(-2xy) \qquad +2x^2y^2(x^2-y^2),$$

which is $= 6x^2y^2z^2-y^2z^4-y^4z^2-z^2x^4-z^4x^2-x^4y^2-x^2y^4.$

The coefficient of $\sqrt{Y}$ is

$$= [x(y^2-z^2)(-3x^2z+z^3)+yz(y^2-z^2)2xy]X$$
$$+y(z^2-x^2)\tfrac{1}{2}X'.(y^3z-yz^3)-zx(z^2-x^2)(y^2-z^2)(X+\tfrac{1}{2}X'x)$$
$$=-2xz(x^2-y^2)(y^2-z^2)X-z(x^2-y^2)(y^2-z^2)(z^2-x^2)\tfrac{1}{2}X'$$
$$=-(x^2-y^2)(y^2-z^2)z\{2xX+\tfrac{1}{2}(z^2-x^2)X'\},$$

where the term in { } is

$$= 2x(1-lx^2+x^4)+(z^2-x^2)(-lx+2x^3),$$
$$= x\{2-l(z^2+x^2)+2z^2x^2\},$$

or the whole coefficient is

$$=-(x^2-y^2)(y^2-z^2)zx\{2-l(z^2+x^2)+2z^2x^2\}.$$

We obtain in like manner the coefficient of $\sqrt{Z}$, and with a little more trouble that of $\sqrt{X}$; and the final result is

$$\begin{aligned}\Omega\,(x^3,\ x,\ \sqrt{X})^2 = &-(z^2-x^2)(x^2-y^2)\,yz\,\{2-l\,(y^2+z^2)+2y^2z^2\}\,\sqrt{X}\\ &-(x^2-y^2)(y^2-z^2)\,zx\,\{2-l\,(z^2+x^2)+2z^2x^2\}\,\sqrt{Y}\\ &-(y^2-z^2)(z^2-x^2)\,xy\,\{2-l\,(x^2+y^2)+2x^2y^2\}\,\sqrt{Z}\\ &+(6x^2y^2z^2-y^2z^4-y^4z^2-z^2x^4-z^4x^2-x^2y^4-x^4y^2)\,\sqrt{X}\sqrt{Y}\sqrt{Z}.\end{aligned}$$

And inasmuch as the equation is a solution of

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}+\frac{dw}{\sqrt{W}}=0,$$

it follows that $\Omega=-\sqrt{W}$, viz., that $\sqrt{W}$ is by the foregoing equation expressed as a function of x, y, z.

The equation $(x^3, x, x^2\sqrt{X}, \sqrt{X}) = 0$, that is,

$$\begin{vmatrix} x^3, & x, & x^2\sqrt{X}, & \sqrt{X} \\ y^3, & y, & y^2\sqrt{Y}, & \sqrt{Y} \\ z^3, & z, & z^2\sqrt{Z}, & \sqrt{Z} \\ w^3, & w, & w^2\sqrt{W}, & \sqrt{W} \end{vmatrix} = 0,$$

gives

$$w=\frac{(x^2,\ 1,\ x\sqrt{X})}{(x^3,\ x,\ \sqrt{X})},$$

where the numerator and denominator are determinants formed with the x, y, z.

Writing $\frac{1}{w}$ for w, it follows that the equation

$$\begin{vmatrix} x^3, & x, & x^2\sqrt{X}, & \sqrt{X} \\ y^3, & y, & y^2\sqrt{Y}, & \sqrt{Y} \\ z^3, & z, & z^2\sqrt{Z}, & \sqrt{Z} \\ w, & w^3, & \sqrt{W}, & w^2\sqrt{W} \end{vmatrix} = 0$$

gives

$$w=\frac{(x^3,\ x,\ \sqrt{X})}{(x^2,\ 1,\ x\sqrt{X})},$$

which last is a transformation of

$$\begin{vmatrix} x^4, & x^2, & 1, & x\sqrt{X} \\ y^4, & y^2, & 1, & y\sqrt{Y} \\ z^4, & z^2, & 1, & z\sqrt{Z} \\ w^4, & w^2, & 1, & w\sqrt{W} \end{vmatrix} = 0.$$

The two equations involving these determinants of the order 4 are consequently equivalent equations.

On the Numerical Value of a Certain Series.

By J. W. L. Glaisher, M.A., F.R.S.

[*Read February 8th*, 1877.]

1. It is known that, if n be uneven, the series

$$\frac{1}{1^n}-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\&c. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

admits of finite expression as a series of terms involving the first n Bernoullian numbers, and having π^n as a factor; but when n is even there is no such formula. If therefore the numerical value of the series (1) be required when n is even, there is apparently no method of obtaining it except from the series itself, or from some other infinite series into which it may admit of being transformed.

The most troublesome case to calculate directly is that of $n=2$, as the series

$$1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\frac{1}{81}-\&c. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

converges very slowly; and the calculation is not very easy even when recourse is had to Euler's formula,

$$\Sigma u_x = \text{const.}+\int u_x\,dx-\tfrac{1}{2}u_x+\frac{B_1}{1.2}\frac{du_x}{dx}-\frac{B_2}{1.2.3.4}\frac{d^3u_x}{dx^3}+\&c.$$

(see § 5); but I have recently obtained the value of the series (2) to 20 places of decimals in the following manner.

2. We know that, if z lie between 1 and -1,

$$\text{arc tan}\,z = z-\frac{z^3}{3}+\frac{z^5}{5}-\frac{z^7}{7}+\frac{z^9}{9}-\&c.,$$

whence, putting $z=\tan t$;

$$t=\tan t-\tfrac{1}{3}\tan^3 t+\tfrac{1}{5}\tan^5 t-\tfrac{1}{7}\tan^7 t+\&c.;$$

and, dividing by $\sin t\cos t$,

$$\frac{t}{\sin t\cos t}=\sec^2 t-\tfrac{1}{3}\tan^2 t\sec^2 t+\tfrac{1}{5}\tan^4 t\sec^2 t-\&c.;$$

whence $\displaystyle\int_0^x \frac{t}{\sin t\cos t}\,dt=\tan x-\tfrac{1}{9}\tan^3 x+\tfrac{1}{25}\tan^5 x-\&c.,$

and the integral $\displaystyle=\int_0^x \frac{2t}{\sin 2t}\,dt=\tfrac{1}{2}\int_0^{2x}\frac{t}{\sin t}\,dt;$

so that $\displaystyle\tfrac{1}{2}\int_0^{2x}\frac{t}{\sin t}\,dt=\tan x-\tfrac{1}{9}\tan^3 x+\tfrac{1}{25}\tan^5 x-\&c. \quad \ldots\ldots \quad (3).$

Now $\displaystyle\text{cosec}\,t=\frac{1}{t}-\frac{1}{t-\pi}-\frac{1}{t+\pi}+\frac{1}{t-2\pi}+\frac{1}{t+2\pi}-\&c.,$

whence $$t \operatorname{cosec} t = 1 - \frac{2t^2}{t^2-\pi^2} + \frac{2t^2}{t^2-(2\pi)^2} - \frac{2t^2}{t^2-(3\pi)^2} + \&c.$$

$$= 1 + \frac{2}{\pi^2}\left(t^2 + \frac{t^4}{\pi^2} + \frac{t^6}{\pi^4} + \frac{t^8}{\pi^6} + \&c.\right)$$

$$- \frac{2}{2^2\pi^2}\left(t^2 + \frac{t^4}{2^2\pi^2} + \frac{t^6}{2^4\pi^4} + \frac{t^8}{2^6\pi^6} + \&c.\right)$$

$$+ \frac{2}{3^2\pi^2}\left(t^2 + \frac{t^4}{3^2\pi^2} + \frac{t^6}{3^4\pi^4} + \frac{t^8}{3^6\pi^6} + \&c.\right)$$

$$- \&c.$$

$$= 1 + 2s_2\frac{t^2}{\pi^2} + 2s_4\frac{t^4}{\pi^4} + 2s_6\frac{t^6}{\pi^6} + 2s_8\frac{t^8}{\pi^8} + \&c.,$$

where $$s_{2n} = 1 - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \frac{1}{4^{2n}} + \frac{1}{5^{2n}} - \&c.;$$

and, by integration,

$$\tfrac{1}{2}\int_0^{2x} \frac{t}{\sin t}\,dt = x + \frac{1}{3}\frac{s_2}{\pi^2}(2x)^3 + \frac{1}{5}\frac{s_4}{\pi^4}(2x)^5 + \frac{1}{7}\frac{s_6}{\pi^6}(2x)^7 + \&c. \quad \ldots(4);$$

whence, from (3) and (4), on writing $\frac{1}{2}\pi x$ for x,

$$\tan \tfrac{1}{2}\pi x - \tfrac{1}{9}\tan^3 \tfrac{1}{2}\pi x + \tfrac{1}{2\,5}\tan^5 \tfrac{1}{2}\pi x - \&c.$$

$$= \tfrac{1}{2}\pi\left(x + \tfrac{2}{3}s_2 x^3 + \tfrac{2}{5}s_4 x^5 + \tfrac{2}{7}s_6 x^7 + \&c.\right) \quad \ldots\ldots\ldots(5),$$

this equation being true if x does not lie beyond the limits $\frac{1}{2}$ and $-\frac{1}{2}$.

In (5) put $x = \frac{1}{2}$, and we find that

$$1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \frac{1}{81} - \&c.$$

$$= \tfrac{1}{2}\pi\left(\frac{1}{2} + \frac{1}{3}\frac{s_2}{2^2} + \frac{1}{5}\frac{s_4}{2^4} + \frac{1}{7}\frac{s_6}{2^6} + \&c.\right).$$

3. If we write $$s'_{2n} = \frac{1}{2^{2n}} - \frac{1}{3^{2n}} + \frac{1}{4^{2n}} - \frac{1}{5^{2n}} + \&c.,$$

so that $$s'_{2n} = 1 - s_{2n},$$

then $$\frac{1}{2} + \frac{1}{3}\frac{s_2}{2^2} + \frac{1}{5}\frac{s_4}{2^4} + \frac{1}{7}\frac{s_6}{2^6} + \&c.$$

$$= \frac{1}{2} + \frac{1}{3}\frac{1}{2^2} + \frac{1}{5}\frac{1}{2^4} + \frac{1}{7}\frac{1}{2^6} + \&c.$$

$$- \frac{1}{3}\frac{s'_2}{2^2} - \frac{1}{5}\frac{s'_4}{2^4} - \frac{1}{7}\frac{s'_6}{2^6} - \&c.$$

$$= -\frac{1}{2} + \log\frac{1+\frac{1}{2}}{1-\frac{1}{2}} - \frac{1}{3}\frac{s'_2}{2^2} - \frac{1}{5}\frac{s'_4}{2^4} - \&c.$$

$$= -\frac{1}{2} + \log 3 - \frac{1}{3}\frac{s'_2}{2^2} - \frac{1}{5}\frac{s'_4}{2^4} - \frac{1}{7}\frac{s'_6}{2^6} - \&c. \quad \ldots.(6).$$

Now the values of $s_2, s_4, \ldots s_{12}$, to 22 places, were contained in a paper* communicated to the Society at the last meeting (January 11, 1877); and on p. 56 of t. iv. of the *Proceedings*, the values of $s_{14}, \ldots s_{34}$ are given to 16 places. From these the values of $s'_2, s'_4, \ldots s'_{32}$ are at once derived; and by means of them the value of the series in (6) can be calculated to 20 places. For the term involving s'_{14} (which is that from which the smallest number of decimals is obtainable) is

$$0{\cdot}00006\quad 08296\quad 54020\quad 3 \div (2\quad 45760),$$

so that 21 places of the value of this term should be correct.

On performing the divisions the values of the terms are found to be as below:

0 . 01479	44138	81323	89848	03	(s_2)
66	20896	31284	42603	03	(s_4)
3	22520	28353	93949	98	(s_6)
	16349	81834	78823	82	(s_8)
	852	70987	36935	67	(s_{10})
	45	50684	64809	59	(s_{12})
	2	47516	49585	08	(s_{14})
		13675	27240	88	(s_{16})
		765	37491	45	(s_{18})
		43	29628	80	(s_{20})
		2	47112	33	(s_{22})
			14210	01	(s_{24})
			822	37	(s_{26})
			47	85	(s_{28})
			2	80	(s_{30})
				16	(s_{32})
				1	(s_{34})
0 . 01549	04806	06472	13111	86	

where a line marked (s_{2n}) contains the value of the term

$$\frac{1}{2n+1}\,\frac{s'_{2n}}{2^{2n}}.$$

The values are given to 22 places, but, as stated above, that of $\frac{1}{15}\,\frac{s'_{14}}{2^{14}}$ is not necessarily correct to more than 21 places.

We know that

$$\log 3 = 1\,.\,09861\quad 22886\quad 68109\quad 69139\quad 52\ldots,$$

* Page 145 of the present volume.

so that we obtain for the expression in (6) the value

$$0.58312\ \ 18080\ \ 61637\ \ 56027\ \ 66\ \ldots \qquad \ldots(7);$$

and, therefore, rejecting the last two figures, we find that, to 20 places,

$$\frac{2}{\pi}\left(1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\frac{1}{81}-\&c.\right) = 0.58312\ \ 18080\ \ 61637\ \ 56028. \qquad \ldots(8).$$

On multiplying (7) by $\frac{1}{2}\pi$, and retaining 20 figures, the value of the series (2) is obtained, viz., we have

$$1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\frac{1}{81}-\&c. = 0.91596\ \ 55941\ \ 77219\ \ 01505\ \ldots$$

4. Also, since

$$1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\frac{1}{81}+\&c. = \tfrac{1}{8}\pi^2 = 1.23370\ \ 05501\ \ 36169\ \ 82735\ \ldots,$$

we find that, to 20 places,

$$\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{9^2}+\frac{1}{13^2}+\&c. = 1.07483\ \ 30721\ \ 56694\ \ 42120,$$

$$\frac{1}{3^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{15^2}+\&c. = 0.15886\ \ 74779\ \ 79475\ \ 40615.$$

5. In the "Messenger of Mathematics," t. vi., pp. 75, 76, I have given ten-place values of these two series, calculated from the formulæ

$$\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{9^2}\ldots+\frac{1}{(4x-3)^2} = C_1-\frac{1}{4}\cdot\frac{1}{4x+1}-\frac{1}{2}\cdot\frac{1}{(4x+1)^2} - \frac{2}{3}\cdot\frac{1}{(4x+1)^3}+\frac{32}{15}\cdot\frac{1}{(4x+1)^5}-\&c.\ldots.(9),$$

$$\frac{1}{3^2}+\frac{1}{7^2}+\frac{1}{11^2}\ldots+\frac{1}{(4x-1)^2} = C_2-\frac{1}{4}\cdot\frac{1}{4x+3}-\frac{1}{2}\cdot\frac{1}{(4x+3)^2} - \frac{2}{3}\cdot\frac{1}{(4x+3)^3}+\&c.\ldots\ldots\ldots(10).$$

Each was the result of a quadruple calculation, viz., putting in (9) $4x+1 = 101, 201, 301, 317$ respectively, the four values found for C_1 were

1.07483 3072, 1.07483 3073, 1.07483 3072, 1.07483 3071;

and putting, in (10), $4x+3 = 103, 203, 303, 319$ respectively, the four

values found for C_2 were

0.15886 7479, 0.15886 7479, 0.15886 7478, 0.15886 7478.

The values adopted for the series were:

$$\frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{9^2} + \&c.\ ad\ inf. = 1.07483\ 3072,$$

$$\frac{1}{3^2} + \frac{1}{7^2} + \frac{1}{11^2} + \&c.\ ad\ inf. = 0.15886\ 7478;$$

and these are seen, on comparing them with the twenty-place values given above in § 4, to be correct to the last place, as also is the value of the series (2) which was given as 0.91596 5594.

6. In the paper in the "Messenger" cited in the last section, it is shown that

$$\frac{3^3 . 7^7 . 11^{11} \ldots (4n-1)^{4n-1}}{1^1 . 5^5 . 9^9 \ldots (4n-3)^{4n-3}} = (4n+1)^{2n} e^{-\frac{1}{2}+\frac{2}{\pi}\left(1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\&c.\right)} \ldots (11),$$

and it was in order to obtain the value of the constant in this formula that I was led to calculate the value of the series

$$1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \&c.$$

We may write (11) in the slightly more convenient form

$$\frac{3^3 . 7^7 . 11^{11} \ldots (4n-1)^{4n-1}}{1^1 . 5^5 . 9^9 \ldots (4n-3)^{4n-3}} = (4n)^{2n} e^{\frac{2}{\pi}\left(1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\&c.\right)},$$

which, using the value in (7),

$$= (4n)^{2n} e^{0.58312\ 18080\ 61637\ 56027 \ldots}$$

7. In connexion with the formulæ in § 2 it may be noted (see "Messenger," *loc. cit.*) that

$$\int_0^x \frac{t}{\sin t}\, dt = \sin x + \frac{2}{3}\frac{\sin^3 x}{3} + \frac{2 . 4}{3 . 5}\frac{\sin^5 x}{5} + \&c.,$$

so that each side of the identical equation (5) is also

$$= \frac{1}{2}\left(\sin \pi x + \frac{2}{3}\frac{\sin^3 \pi x}{3} + \frac{2 . 4}{3 . 5}\frac{\sin^5 \pi x}{5} + \&c.\right)$$

On the Area of the Quadrangle formed by the Four Points of Intersection of Two Conics. By C. Leudesdorf, M.A.

[*Read February 8th*, 1877.]

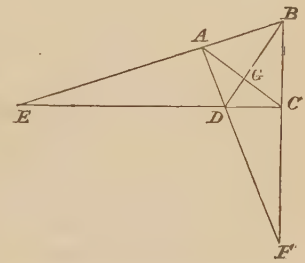
Let the conics be

$$S = (a, b, c, f, g, h\mathfrak{X}x, y, 1)^2 = 0,$$

and

$$S' = (a', b', c', f', g', h'\mathfrak{X}x, y, 1)^2 = 0,$$

and write A, B, C, F, G, H for $bc-f^2$, $ca-g^2$, $ab-h^2$, $gh-af$, $hf-bg$, $fg-ch$, as usual. We suppose that the four points of intersection are all real: so that the discriminant of the cubic $\kappa^3\Delta+\kappa^2\Theta+\kappa\Theta'+\Delta'=0$ is negative. We suppose also that the quadrangle formed by these points is non-reentrant; the condition for this may be found from the consideration that no real ellipse or parabola can be drawn through four points which form a reentrant quadrangle, and is $\nu^2-4CC'>0$, where ν stands for $ab'+ba'-2hh'$. Before considering the general problem, we must investigate the simpler case in which $S=0$ and $S'=0$ each represent a pair of right lines. Taking any non-reentrant quadrangle, as ABCD, whose opposite sides intersect in the points E, F, G; we see that, of these points, two will always lie without the quadrangle (as E and F), and the third within it (as G).



Considering first the quadrangle ABCD as formed by the points of intersection of the two pairs of straight lines AB, CD ($S=0$) and AD, BC ($S'=0$), whose vertices E, F both lie *without* it, the area ABCD may be found as the difference of the triangles cut off from the pair of lines $S=0$ by each of the lines of the pair $S'=0$; *i.e.*, as $\Delta EBC-\Delta EAD$. We require then, in the first place, an expression for the area of the triangle cut off from the pair of straight lines $S=0$ by another straight line, as $lx+my+n=0$.

Assume

$$S = (px+qy+r)\left(\frac{a}{p}x+\frac{b}{q}y+\frac{c}{r}\right),$$

so that

$$c\frac{q}{r}+b\frac{r}{q}=2f, \text{ \&c. \&c.};$$

and therefore

$$c\frac{q}{r}-b\frac{r}{q}=2\sqrt{-A}=2\frac{\sqrt{AC}}{\sqrt{-C}}=\frac{2G}{\sqrt{-C}}, \text{ \&c. \&c.}$$

Then the area of the triangle required is

$$\left|\begin{matrix} l & m & n \\ p & q & r \\ \frac{a}{p} & \frac{b}{q} & \frac{c}{r} \end{matrix}\right|^2 \div 2 \left|\begin{matrix} l & m \\ p & q \end{matrix}\right| \left|\begin{matrix} p & q \\ \frac{a}{p} & \frac{b}{q} \end{matrix}\right| \left|\begin{matrix} \frac{a}{p} & \frac{b}{q} \\ l & m \end{matrix}\right|$$

$$= \frac{\left\{ l\left(c\frac{q}{r} - b\frac{r}{q}\right) + m\left(a\frac{r}{p} - c\frac{p}{r}\right) + n\left(b\frac{p}{q} - a\frac{q}{p}\right)\right\}^2}{2\left(b\frac{p}{q} - a\frac{q}{p}\right)\left\{ bl^2 + am^2 - lm\left(b\frac{p}{q} - a\frac{q}{p}\right)\right\}}$$

$$= \frac{1}{(-\mathrm{C})^{\frac{3}{2}}} \frac{(l\mathrm{G} + m\mathrm{F} + n\mathrm{C})^2}{bl^2 + am^2 - 2hlm} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let now $\left(x - \frac{\mathrm{G}'}{\mathrm{C}'}\right)\sin\theta - \left(y - \frac{\mathrm{F}'}{\mathrm{C}'}\right)\cos\theta = 0$ be the equation of one of the straight lines composing S′; so that we have the condition $a'\cos^2\theta + b'\sin^2\theta + 2h'\cos\theta\sin\theta = 0$, or

$$b'\tan^2\theta + 2h'\tan\theta + a' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The area of the triangle cut off by this straight line from the pair S=0 is found, by writing in (1)

$$l = \sin\theta, \quad m = -\cos\theta, \quad n = -\frac{\mathrm{G}'}{\mathrm{C}'}\sin\theta + \frac{\mathrm{F}'}{\mathrm{C}'}\cos\theta,$$

to be

$$\frac{1}{(-\mathrm{C})^{\frac{3}{2}}} \frac{\left\{ \mathrm{G}\sin\theta - \mathrm{F}\cos\theta + \frac{\mathrm{C}}{\mathrm{C}'}(\mathrm{F}'\cos\theta - \mathrm{G}'\sin\theta)\right\}^2}{a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta}$$

or

$$\frac{1}{(-\mathrm{C})^{\frac{3}{2}}\mathrm{C}'^2} \frac{\{(\mathrm{GC}' - \mathrm{G}'\mathrm{C})\tan\theta - (\mathrm{FC}' - \mathrm{F}'\mathrm{C})\}^2}{b\tan^2\theta + 2h\tan\theta + a}.$$

If then θ_1, θ_2 be the two values of θ which satisfy (2),

$$\text{Area ABCD} = \Delta\mathrm{EBC} - \Delta\mathrm{EAD}$$

$$= \frac{1}{(-\mathrm{C})^{\frac{3}{2}}\mathrm{C}'^2}\left[\frac{\{(\mathrm{GC}' - \mathrm{G}'\mathrm{C})\tan\theta_1 - (\mathrm{FC}' - \mathrm{F}'\mathrm{C})\}^2}{b\tan^2\theta_1 + 2h\tan\theta_1 + a} - \frac{\{(\mathrm{GC}' - \mathrm{G}'\mathrm{C})\tan\theta_2 - (\mathrm{FC}' - \mathrm{F}'\mathrm{C})\}^2}{b\tan^2\theta_2 + 2h\tan\theta_2 + a}\right]$$

$$= \frac{1}{(-\mathrm{C})^{\frac{3}{2}}\mathrm{C}'^2}(\tan\theta_1 - \tan\theta_2) \frac{\left\{\begin{matrix} (\mathrm{GC}' - \mathrm{G}'\mathrm{C})^2\{2h\tan\theta_1\tan\theta_2 + a(\tan\theta_1 + \tan\theta \\ +(\mathrm{FC}' - \mathrm{F}'\mathrm{C})^2\{-b(\tan\theta_1 + \tan\theta_2) - 2h\} \\ +2(\mathrm{GC}' - \mathrm{G}'\mathrm{C})(\mathrm{FC}' - \mathrm{F}'\mathrm{C})\{b\tan\theta_1\tan\theta_2 - a\} \end{matrix}\right.}{(b\tan^2\theta_1 + 2h\tan\theta_1 + a)(b\tan^2\theta_2 + 2h\tan\theta_2 + a}$$

which, by means of equation (2), reduces to

$$\frac{4}{(\mathrm{CC}')^{\frac{3}{2}}(\nu^2 - 4\mathrm{CC}')}[(\mathrm{GC}' - \mathrm{G}'\mathrm{C})^2(ha' - h'a) + (\mathrm{FC}' - \mathrm{F}'\mathrm{C})^2(h'b - hb') + (\mathrm{GC}' - \mathrm{G}'\mathrm{C})(\mathrm{FC}' - \mathrm{F}'\mathrm{C})(a'b - ab')].$$

The expression within the brackets may be written

$$\{a'(\mathrm{GC'}-\mathrm{G'C})+h'(\mathrm{FC'}-\mathrm{F'C})\}\{h(\mathrm{GC'}-\mathrm{G'C})+b(\mathrm{FC'}-\mathrm{F'C})\}$$
$$-\{a(\mathrm{GC'}-\mathrm{G'C})+h(\mathrm{FC'}-\mathrm{F'C})\}\{h'(\mathrm{GC'}-\mathrm{G'C})+b'(\mathrm{FC'}-\mathrm{F'C})\}$$
$$=\{\mathrm{C'}(a'\mathrm{G}+h'\mathrm{F}+g'\mathrm{C})-\mathrm{C}(a'\mathrm{G'}+h'\mathrm{F'}+g'\mathrm{C'})\}$$
$$\times\{\mathrm{C'}(h\mathrm{G}+b\mathrm{F}+f\mathrm{C})-\mathrm{C}(h\mathrm{G'}+b\mathrm{F'}+f\mathrm{C'})\}$$
$$-\{\mathrm{C'}(a\mathrm{G}+h\mathrm{F}+g\mathrm{C})-\mathrm{C}(a\mathrm{G'}+h\mathrm{F'}+g\mathrm{C'})\}$$
$$\times\{\mathrm{C'}(h'\mathrm{G}+b'\mathrm{F}+f'\mathrm{C})-\mathrm{C}(h'\mathrm{G'}+b'\mathrm{F'}+f'\mathrm{C'})\}$$
$$=\mathrm{CC'}\{(a\mathrm{G'}+h\mathrm{F'}+g\mathrm{C'})(h'\mathrm{G}+b'\mathrm{F}+f'\mathrm{C})$$
$$-(a'\mathrm{G}+h'\mathrm{F}+g'\mathrm{C})(h\mathrm{G'}+b\mathrm{F'}+f\mathrm{C'})\}\ldots\ldots(3),$$

and this may be proved * to be equal to $\sqrt{(\mathrm{CC'})^3(\nu\Theta\Theta'-\mathrm{C}\Theta'^2-\mathrm{C'}\Theta^2)}$; thus the area of the quadrangle ABCD

$$=\frac{4}{\nu^2-4\mathrm{CC'}}\sqrt{\nu\Theta\Theta'-\mathrm{C}\Theta'^2-\mathrm{C'}\Theta^2}\quad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Let us next consider the quadrangle ABCD as formed by the points of intersection of two pairs of straight lines whose vertices lie the one *without*, the other *within*, the quadrangle, such as AB, CD and AC, BD. Let the equation to the former pair be $\mathrm{S}=0$, and to the latter $\mathrm{S'}=0$. If we were to apply the formula (4), just found, to this case, it would clearly give us, not the area ABCD, but the difference of the triangles cut off from the pair of straight lines $\mathrm{S}=0$ by each of the straight lines which compose the pair $\mathrm{S'}=0$; *i. e.*, $\Delta\mathrm{EAC}\sim\Delta\mathrm{EBD}$, or $\Delta\mathrm{GCD}\sim\Delta\mathrm{GAB}$. Thus a different formula must be found for this case; and this may be done by adapting the formula (4) to the case of the two pairs of straight lines AB, CD ($\mathrm{S}=0$), and AD, BC (whose equation is evidently $\Theta'\mathrm{S}-\Theta\mathrm{S'}=0$), which have both their vertices lying without the quadrangle, and to which therefore the investigation above applies. We have only to write, instead of a', b', c', f', g', h', $a'-\frac{\Theta'}{\Theta}a$, $b'-\frac{\Theta'}{\Theta}b$, &c. When this is done it is found that ν must be replaced by $\nu-2\mathrm{C}\frac{\Theta'}{\Theta}$, and C' by $\frac{\mathrm{C}\Theta'^2+\mathrm{C'}\Theta^2-\nu\Theta\Theta'}{\Theta^2}$; for Θ' we must

* The work of direct proof is omitted, being rather tedious. The identity stated may be seen, however, to be probably true, since the expression within the brackets in (3) and the quantity $\sqrt{\mathrm{CC'}(\nu\Theta\Theta'-\mathrm{C}\Theta'^2-\mathrm{C}\Theta^2)}$ denote, when equated to zero, exactly the same thing, viz., that one of the pairs of straight lines (E), (F), (G) should consist of parallel lines; and the dimensions of the two are the same. In the more general case, where $\mathrm{S}=0$, $\mathrm{S'}=0$ denote any two conics, the expression within the brackets in (3) is equal to the square root of the determinant

$$\begin{vmatrix} 0 & \Delta & \Theta & \Theta' & \Delta' \\ \Delta & \Theta & \Theta' & \Delta' & 0 \\ \mathrm{C} & \nu & \mathrm{C'} & 0 & 0 \\ 0 & \mathrm{C} & \nu & \mathrm{C'} & 0 \\ 0 & 0 & \mathrm{C} & \nu & \mathrm{C'} \end{vmatrix}$$

write $a\left[\frac{\Theta'^2}{\Theta^2}A+A'-\frac{\Theta'}{\Theta}(bc'+b'c-2ff')\right]+\&c.$, which reduces to $-\Theta'$; while C, Θ, ν^2-4CC' remain unchanged. Thus the formula (4) will be changed into

$$\frac{4}{\nu^2-4CC'}\sqrt{\left(-\nu+2C\frac{\Theta'}{\Theta}\right)\Theta\Theta'-C\Theta'^2-C'\Theta^2-C\Theta'^2+\nu\Theta\Theta'},$$

that is, into

$$\frac{4\Theta\sqrt{-C'}}{\nu^2-4CC'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

which is therefore the expression for the area ABCD in this case.

Lastly, considering the quadrangle ABCD as formed by the two pairs of straight lines AC, BD ($S=0$), and AD, BC ($S'=0$), where again one vertex lies without and the other within the quadrangle, it is clear from the foregoing that the area ABCD will in this case be represented by the expression

$$\frac{4\Theta'\sqrt{-C}}{\nu^2-4CC'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

If now, in this last case, we had written down the formula (4), it would have given us, not the area ABCD, but the difference of the triangles cut off from the pair of straight lines $S=0$ by each of the straight lines composing the pair AD, BC; *i. e.*, $\Delta GAD \sim \Delta GBC$. And the formula (5) would in this case have given us the difference of the triangles cut off from the pair of straight lines $S=0$ by each of those composing the third pair AB, CD; *i. e.*, $\Delta GCD \sim \Delta GAB$. Similar reasoning will apply to each of the other two cases; the roots of the cubic

$$\left(x-\frac{4\Theta\sqrt{-C}}{\nu^2-4CC'}\right)\left(x-\frac{4\Theta\sqrt{-C'}}{\nu^2-4CC'}\right)\left(x-\frac{4\sqrt{\nu\Theta\Theta'-C\Theta'^2-C'\Theta^2}}{\nu^2-4CC'}\right)=0 \ldots(7),$$

being, in each case, the three quantities

area ABCD, $\Delta GAD \sim \Delta GBC$, $\Delta GCD \sim \Delta GAB$.

But of these the area ABCD must always be the greatest. Therefore, when $S=0$, $S'=0$ denote each a pair of straight lines, the area of the quadrangle formed by their four points of intersection is the greatest root of the cubic (7).

Let us now proceed to consider the general case, in which $S=0$, $S'=0$ denote each a conic of any kind. Let κ_1, κ_2, κ_3 be the three values of κ for which $\kappa S+S'=0$ breaks up into straight lines; and suppose κ_3 that value for which the vertex of the pair of lines $\kappa S+S'=0$ lies *within* the quadrangle formed by the four points of intersection of S and S'. Then the area of this quadrangle may be deduced from the formula (4), applied to the case of the two pairs $\kappa_1 S+S'=0$, $\kappa_2 S+S'=0$. We must, in order to do this, write instead

of a, b, &c., $\kappa_1 a+a'$, $\kappa_1 b+b'$, &c.; and instead of a', b', &c., $\kappa_2 a+a'$, $\kappa_2 b+b'$, &c. We must therefore write, for A, $(\kappa_1 b+b')(\kappa_1 c+c')-(\kappa_1 f+f')^2$, &c. &c.; for F, $(\kappa_1 g+g')(\kappa_1 h+h')-(\kappa_1 a+a')(\kappa_1 f+f')$, &c. &c.; *i. e.*, if we put

$$\lambda = bc'+b'c-2ff', \qquad \mu = ca'+c'a-2gg', \qquad \nu = ab'+a'b-2hh',$$
$$\lambda' = gh'+g'h-af'-a'f, \quad \mu' = hf'+h'f-bg'-b'g, \quad \nu' = fg'+f'g-ch'-c'h$$

we must write, instead of A, B, C, F, G, H respectively,

$$\kappa_1^2 \mathrm{A}+\kappa_1 \lambda + \mathrm{A}', \quad \kappa_1^2 \mathrm{B}+\kappa_1 \mu + \mathrm{B}', \quad \kappa_1^2 \mathrm{C}+\kappa_1 \nu + \mathrm{C}',$$
$$\kappa_1^2 \mathrm{F}+\kappa_1 \lambda' + \mathrm{F}', \quad \kappa_1^2 \mathrm{G}+\kappa_1 \mu' + \mathrm{G}', \quad \kappa_1^2 \mathrm{H}+\kappa_1 \nu' + \mathrm{H}',$$

and for A', B', C', F', G', H' precisely similar quantities, κ_2 being substituted for κ_1. It is easily seen that ν will be replaced by $2\kappa_1\kappa_2 \mathrm{C}+(\kappa_1+\kappa_2)\nu+2\mathrm{C}'$, while $\nu^2-4\mathrm{CC}'$ becomes $(\kappa_1-\kappa_2)^2(\nu^2-4\mathrm{CC}')$. For Θ we must put

$$\begin{aligned}
&\Sigma(\kappa_2 a+a')(\kappa_1^2 \mathrm{A}+\kappa_1\lambda+\mathrm{A}') \\
&= \kappa_1^2\kappa_2 \Sigma a\mathrm{A}+\kappa_1\kappa_2\Sigma a\lambda+\kappa_2\Sigma a\mathrm{A}'+\kappa_1^2\Sigma a'\mathrm{A}+\kappa_1\Sigma a'\lambda+\Sigma a'\mathrm{A}' \\
&= 3\kappa_1^2\kappa_2\Delta+2\kappa_1\kappa_2\Theta+\kappa_2\Theta'+\kappa_1^2\Theta+2\kappa_1\Theta'+3\Delta' \\
&= \Delta\{3\kappa_1^2\kappa_2-(2\kappa_1\kappa_2+\kappa_1^2)(\kappa_1+\kappa_2+\kappa_3)+(\kappa_2+2\kappa_1)(\kappa_1\kappa_2+\kappa_2\kappa_3+\kappa_3\kappa_1) \\
&\qquad -3\kappa_1\kappa_2\kappa_3\}
\end{aligned}$$

(from the equation $\kappa^3\Delta+\kappa^2\Theta+\kappa\Theta'+\Delta'=0$, satisfied by κ_1, κ_2, κ_3)

$$= \Delta(\kappa_1-\kappa_2)^2(\kappa_3-\kappa_1).$$

Similarly, Θ' will be replaced by $\Delta(\kappa_1-\kappa_2)^2(\kappa_3-\kappa_2)$. Making these substitutions in (4), we find for the required area

$$\frac{4\Delta}{\nu^2-4\mathrm{CC}'}[-(\kappa_1^2\mathrm{C}+\kappa_1\nu+\mathrm{C}')(\kappa_3-\kappa_2)^2-(\kappa_2^2\mathrm{C}+\kappa_2\nu+\mathrm{C}')(\kappa_3-\kappa_1)^2 + (2\kappa_1\kappa_2\mathrm{C}+\overline{\kappa_1+\kappa_2}\nu+2\mathrm{C}')(\kappa_3-\kappa_2)(\kappa_3-\kappa_1)]^{\frac{1}{2}},$$

which reduces to

$$\frac{4\Delta}{\nu^2-4\mathrm{CC}'}(\kappa_1-\kappa_2)\sqrt{-(\mathrm{C}\kappa_3^2+\nu\kappa_3+\mathrm{C}')} \quad \ldots\ldots\ldots\ldots\ldots (8).$$

Writing x for the area of the quadrangle, we have

$$\left(\frac{\nu^2-4\mathrm{CC}'}{4}x\right)^2 = -\Delta^2(\kappa_1-\kappa_2)^2(\mathrm{C}\kappa_3^2+\nu\kappa_3+\mathrm{C}')$$
$$= -\left(\Theta^2-2\Delta\Theta'-\Delta^2\kappa_3^2+\frac{2\Delta\Delta'}{\kappa_3}\right)(\mathrm{C}\kappa_3^2+\nu\kappa_3+\mathrm{C}'),$$

or

$$\kappa_3^4\Delta\Theta\mathrm{C}+\kappa_3^3\{\mathrm{C}(\Theta^2-2\Delta\Theta')+\nu\Delta\Theta\}+\kappa_3^2\{3\Delta\Delta'\mathrm{C}+\Delta\Theta\mathrm{C}'+\nu(\Theta^2-\Delta\Theta')\}$$
$$+\kappa_3\left\{3\nu\Theta\Theta'+\mathrm{C}'(\Theta^2-\Delta\Theta')+\left(\frac{\nu^2-4\mathrm{CC}'}{4}x\right)^2\right\}+3\mathrm{C}'\Delta\Delta'=0;$$

and by eliminating κ_3 between this equation and $\kappa_3^3\Delta+\kappa_3^2\Theta+\kappa_3\Theta'+\Delta'=0$, we obtain a cubic in x^2; and, just as above, it is seen that the greatest root of this cubic is the square of the area of the quadrangle ABCD; the other roots being $(\Delta GAB \sim \Delta GCD)^2$ and $(\Delta GAD \sim \Delta GBC)^2$. We may form this cubic more conveniently thus:—If x_1^2, x_2^2, x_3^2 be its roots, and if δ stand for $CC'-\frac{\nu^2}{4}$,

$$-\delta^2(x_1^2+x_2^2+x_3^2) = \Delta^2(\kappa_1-\kappa_2)^2(C\kappa_3^2+\nu\kappa_3+C')+\ldots+\ldots$$
$$= \Delta^2\{C\Sigma\kappa_3^2(\kappa_1-\kappa_2)^2+\nu\Sigma\kappa_3(\kappa_1-\kappa_2)^2+C'\Sigma(\kappa_1-\kappa_2)^2\},$$

which, when the symmetric functions are calculated out, is found to be

$$2C(\Theta'^2-3\Delta'\Theta)-\nu(\Theta\Theta'-9\Delta\Delta')+2C'(\Theta^2-3\Delta\Theta').$$

This is one of the fundamental invariants of the cubic $\kappa^3\Delta+\kappa^2\Theta+\kappa\Theta'+\Delta'$ and the quadratic $C\kappa^2+\nu\kappa+C'=0$. If it be denoted by I, we have $x_1^2+x_2^2+x_3^2 = -\frac{I}{\delta^2}$ .....(9).

Also, by (8),

$$-\delta^6 x_1^2 x_2^2 x_3^2 = \Delta^6\Pi(\kappa_1-\kappa_2)^2\,\Pi(C\kappa_3^2+\nu\kappa_3+C').$$

Now $\Pi(\kappa_1-\kappa_2)^2$ is equal to $-\frac{D}{\Delta^4}$, if D be the discriminant of the cubic $\kappa^3\Delta+\kappa^2\Theta+\kappa\Theta'+\Delta'$; also $\Pi(C\kappa_3^2+\nu\kappa_3+C')$ is equal to $\frac{1}{\Delta^2}$ times the resultant (R) of this cubic and the quadratic $C\kappa^2+\nu\kappa+C'$; so that

$$x_1^2 x_2^2 x_3^2 = \frac{DR}{\delta^6} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10).$$

Again, by (8),

$$\delta^4(x_1^4+x_2^4+x_3^4) = \Delta^4(\kappa_1-\kappa_2)^4(C\kappa_3^2+\nu\kappa_3+C')^2+\ldots+\ldots,$$

which may be reduced, by the calculation of symmetric functions, to

$$2C^2(\Theta'^2-3\Delta'\Theta)^2+2C'^2(\Theta^2-3\Delta\Theta')$$
$$+(\nu^2+2CC')(\Theta^2\Theta'^2+27\Delta^2\Delta'^2-2\Delta\Theta'^3-2\Delta'\Theta^3)$$
$$-2C\nu(\Theta'^2-3\Delta'\Theta)(\Theta\Theta'-9\Delta\Delta')-2C'\nu(\Theta^2-3\Delta\Theta')(\Theta\Theta'-9\Delta\Delta'),$$

which may be written as

$$\tfrac{1}{2}\{2C(\Theta'^2-3\Delta'\Theta)-\nu(\Theta\Theta'-9\Delta\Delta')+2C'(\Theta^2-3\Delta\Theta')^2\}$$
$$-\left(\frac{\nu^2}{2}-2CC'\right)(27\Delta^2\Delta'^2-\Theta^2\Theta'^2+4\Delta\Theta'^3+4\Delta'\Theta^3-18\Delta\Delta'\Theta\Theta'),$$

that is, as $\frac{I^2}{2}+2\delta D$; so that

$$x_1^2x_2^2+x_2^2x_3^2+x_3^2x_1^2 = \tfrac{1}{2}\{(x_1^2+x_2^2+x_3^2)^2-(x_1^4+x_2^4+x_3^4)\}$$
$$= \tfrac{1}{2}\left\{\frac{I^2}{\delta^4}-\frac{1}{\delta^4}\left(\frac{I^2}{2}+2\delta D\right)\right\}, \text{ by (9),}$$
$$= \frac{1}{\delta^4}\left(\frac{I^2}{4}-\delta D\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11).$$

From (9), (10), and (11), the required cubic is seen to be

$$x^6+\frac{\mathrm{I}}{\delta^2}x^4+\frac{\frac{\mathrm{I}^2}{4}-\delta\mathrm{D}}{\delta^4}-\frac{\mathrm{DR}}{\delta^6}=0,$$

or

$$x^2\left(x^2+\frac{\mathrm{I}}{2\delta^2}\right)^2-\frac{\mathrm{D}}{\delta^3}\left(x^2+\frac{\mathrm{R}}{\delta^3}\right)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (12),$$

and the greatest root of this equation is the square of the area of the quadrangle ABCD formed by the four points of intersection of the two conics S and S′; the other roots being $(\Delta\mathrm{GAB}\sim\Delta\mathrm{GCD})^2$ and $(\Delta\mathrm{GAD}\sim\Delta\mathrm{GBC})^2$.

With regard to the four quantities δ, D, R, I, in terms of which this area can therefore be found, these are seen to be the four invariants of the system formed by the cubic $\kappa^3\Delta+\kappa^2\Theta+k\Theta'+\Delta'$ and the quadratic $\mathrm{C}\kappa^2+\nu\kappa+\mathrm{C}'$; viz., the discriminant of the quadratic (δ), that of the cubic (D), their resultant (R), and the intermediate invariant which arises from combining the quadratic with the Hessian of the cubic (I). The cubic and the quadratic are themselves discriminants, the former of $\kappa\mathrm{S}+\mathrm{S}'$, the latter of $\kappa(a, h, b\,\mathrm{X}\,x, y)^2+(a', h', b'\,\mathrm{X}\,x, y)^2$.

We may notice the following particular cases :—

1. When S and S′ both break up into pairs of straight lines, Δ and Δ' vanish, so that

$\mathrm{D}=-\Theta^2\Theta'^2$, $\mathrm{R}=\mathrm{CC}'(\mathrm{C}\Theta'^2-\nu\Theta\Theta'+\mathrm{C}'\Theta^2)$, $\mathrm{I}=2\mathrm{C}\Theta'^2-\nu\Theta\Theta'+2\mathrm{C}'\Theta^2$; and (12) reduces to the form (7), as it should do.

2. When $\mathrm{D}=0$, one root of (12) is zero, the other two being each equal to $-\frac{\mathrm{I}}{2\delta^2}$; *i. e.*, when the conics S, S′ touch, the square root of this is the value of the area of the triangle formed by the three points of intersection. From the figure it is seen that, when two of the four points A, B, C, D (say C, D) coincide, the three roots of (12), $(\mathrm{ABCD})^2$, $(\Delta\mathrm{GAB}\sim\Delta\mathrm{GCD})^2$, $(\Delta\mathrm{GAD}\sim\Delta\mathrm{GBC})^2$ reduce to $(\Delta\mathrm{ABC})^2$, $(\Delta\mathrm{ABC})^2$, 0 respectively.

3. When $\mathrm{R}=0$, one of the roots of (12) is zero, the greater of the other two roots being equal to $\sqrt{\frac{\mathrm{D}}{\delta^3}}-\frac{\mathrm{I}}{2\delta^2}$. But $\mathrm{R}=0$ is the condition that one of the pairs of chords of intersection of the two conics should consist of parallel straight lines; in this case, then, the square root of the above expression gives the area ABCD. From the figure it is seen that, if AB, CD be parallel, $\Delta\mathrm{GAD}=\Delta\mathrm{GBC}$, so that one of the roots of (12) ought to vanish.

4. If $\mathrm{S}=0$ denote a conic, and $\mathrm{S}'=0$ a pair of straight lines whose vertex lies *within* the quadrangle formed by the four points of intersec-

tion of S and S′ (*e. g.*, if S be an ellipse or a parabola, S′ may be any pair of straight lines which meet within the conic), then Δ' vanishes, and one value of κ is zero, the other two being given by the equation $\Delta\kappa^2+\Theta\kappa+\Theta'=0$, so that their difference is $-\frac{\sqrt{\Theta^2-4\Delta\Theta'}}{\Delta}$, and, by (8), the area of the quadrangle formed by the points of intersection is

$$\frac{4\sqrt{-C'}}{\nu^2-4CC'}\sqrt{\Theta^2-4\Delta\Theta'}.$$

5. If, in this last case (4), the pair of straight lines S′ be supposed to meet on S, then also $\Theta'=0$, and the area of the triangle formed is $-\frac{\Theta}{\delta}\sqrt{-C'}$.

6. If $S=0$, $S'=0$ each represent a pair of parallel straight lines, then C, C′ vanish as well as Δ, Δ'; so that, by (7), the area of the parallelogram formed is $\sqrt{\frac{\Theta\Theta'}{\nu^3}}$.

Geometrical Illustration of a Theorem relating to an Irrational Function of an Imaginary Variable. By Prof. Cayley.

[*Read May 11th*, 1876.]*

If we have v, a function of u, determined by an equation $f(u, v)=0$, then to any given imaginary value $x+iy$ of u there belong two or more values, in general imaginary, $x'+iy'$ of v: and for the complete understanding of the relation between the two imaginary variables, we require to know the series of values $x'+iy'$ which correspond to a given series of values $x+iy$, of v, u respectively. We must for this purpose take x, y as the coordinates of a point P in a plane Π, and x', y' as the coordinates of a corresponding point P′ in another plane Π'. The series of values $x+iy$ of u is then represented by means of a curve in the first plane, and the series of values $x'+iy'$ of v by means of a corresponding curve in the second plane. The correspondence between the two points P and P′ is of course established by the two equations into which the given equation $f(x+iy, x'+iy')=0$ breaks up, on the assumption that x, y, x', y' are all of them real. If we assume that the coefficients in the equation are real, then the two equations are

$$f(x+iy, x'+iy')+f(x-iy, x'-iy')=0,$$
$$f(x+iy, x'+iy')-f(x-iy, x'-iy')=0;$$

viz., if in these equations we regard either set of coordinates, say (x, y),

* Not received in time to be placed earlier in the Proceedings.

as constants, then the other set (x', y') are the coordinates of any real point of intersection of the curves represented by these equations respectively.

I consider the particular case where the equation between u, v is $u^2+v^2=a^2$: we have here $(x+iy)^2+(x'+iy')^2=a^2$: so that to a given point P in the first plane there correspond in general two points P_1', P_2' in the second plane: but to each of the points A and B, coordinates $(a, 0)$ and $(-a, 0)$, there corresponds only a single point in the second plane.

We have here a particular case of a well known theorem: viz., if from a given point P we pass by a closed curve not containing within it either of the points A or B, back to the initial point P, we pass in the other plane from P_1' by a closed curve back to P_1'; and similarly from P_2' by a closed curve back to P_2': but if the closed curve described by P contain within it A or B, then, in the other plane, we pass continuously from P_1' to P_2'; and also continuously from P_2' to P_1'.

The relations between (x, y), (x', y') are

$$x'^2-y'^2=a^2-(x^2-y^2),$$
$$x'y'=-xy,$$

whence also $(x'^2+y'^2)=a^4-2a^2(x^2-y^2)+(x^2+y^2)^2$.

And if the point (x, y) describe a curve $x^2+y^2=\phi(x^2-y^2)$, then will the point (x', y') describe a curve $x'^2+y'^2=\psi(x'^2-y'^2)$ obtained by the elimination of x^2-y^2 from the two equations

$$x'^2-y'^2= a^2- (x^2-y^2),$$
$$(x'^2+y'^2)^2= a^4-2a^2(x^2-y^2)+\phi(x^2-y^2),$$

viz., this is $(x'^2+y'^2)^2=-a^4+2a^2(x'^2-y'^2)+\phi\{a^2-(x^2-y^2)\}$.

In particular, if the one curve be $(x^2+y^2)^2=\alpha+\beta(x^2-y^2)$; then the other curve is $(x'^2+y'^2)^2=-a^4+2a^2(x'^2-y'^2)+\alpha+\beta\{a^2-(x'^2-y'^2)\}$,

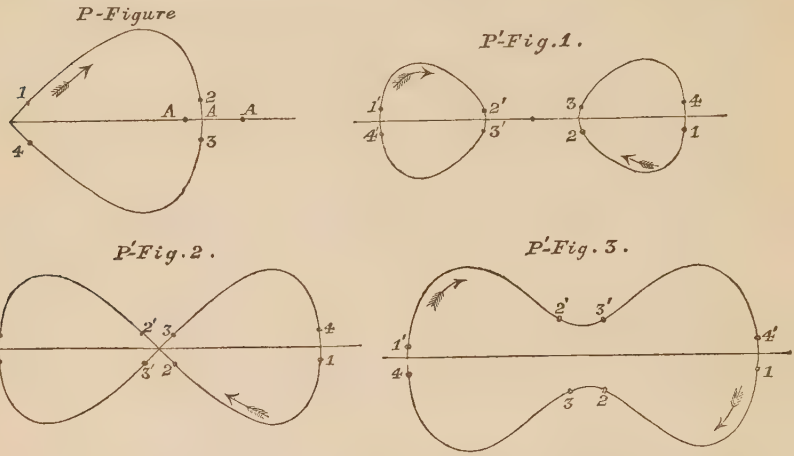
that is, $(x'^2+y'^2)^2=\alpha'+\beta'(x'^2-y'^2)$,

where $\alpha'=-a^4+\beta a^2+\alpha$, $\beta'=2a^2-\beta$;

or, writing for greater simplicity $a=1$, then $\alpha'=-1+\alpha+\beta$, $\beta'=2-\beta$, or, in particular, if $\alpha=0$, then $\alpha'=-1+\beta$, $\beta'=2-\beta$.

Supposing successively $\beta<1$, $\beta=1$, and $\beta>1$, then in each case P describes a closed curve or half figure-of-eight, as shown in the annexed P-figure; but in the first case the point A is inside the curve, in the second case on it, and in the third case outside it, as shown by the letters A, A, A of the figure; and, corresponding to the three cases respectively, we have the three P′-figures, the curve in the first of them consisting of two ovals, in the second of them being a figure of eight, and in the third a twice-indented or pinched oval: the small figures

1, 2, 3, 4 in the P-figure, and 1, 2, 3, 4 and 1′, 2′, 3′, 4′ in the P′-figures serve to show the corresponding positions of the points P and P'_1, P'_2



respectively; and the courses are further indicated by the arrows. And we thus see how the two separate closed curves described by P'_1 and P'_2, as in figure 1, change into the single closed curve described one half of it by P'_1 and the other half of it by P'_2 as in figure 3.

March 8th, 1877.

C. W. MERRIFIELD, Esq., F.R.S., Vice-President, in the Chair.

Mr. C. Pendlebury, B.A., St. John's College, Cambridge, and Mathematical Master in St. Paul's School, was proposed for election; and Mr. R. F. Davis was admitted into the Society.

The following communications were made:—

"On a New View of Pascal's Hexagram": Mr. T. Cotterill.

"On a Class of Integers expressible as the sum of Two Integral Squares": Mr. T. Muir.

"Some Properties of the Double-Theta Functions" (founded on papers by Goepel and Rosenhain): Prof. Cayley.

"A Property of an Envelope": Mr. J. J. Walker.

The following presents were received:

"Elementary Treatise on the Differential Calculus," by B. Williamson, F.R.S., 1877: from the Author.

"Educational Times," March.

On a Class of Integers expressible as the Sum of Two Integral Squares. *By* Thomas Muir, M.A., F.R.S.E.

[*Read March 8th*, 1877.]

It is known from Lagrange, that any integer whose square root, when expressed in the ordinary way as a continued fraction, has an odd number of terms in its cycle of partial denominators, must be the sum of two integral squares: *e.g.*,

$$\sqrt{365} = 19 + \cfrac{1}{\overset{*}{9} + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{9 + \cfrac{1}{\overset{*}{38} + \ddots}}}}}$$

and $$365 = 2^2 + 19^2 = 13^2 + 14^2;$$

$$\sqrt{(2500m^2 + 1500m + 325)} = (50m + 15) + \cfrac{1}{\overset{*}{m} + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{m + \cfrac{1}{\overset{*}{100m + 30} + \ddots}}}}}$$

and $$2500m^2 + 1500m + 325 = (50m + 15)^2 + (10)^2$$
$$\text{or} \quad = (40m + 18)^2 + (30m + 1)^2.$$

Further, it is known† that the general expression for such integers is

$$\text{(A)} \quad \begin{cases} \left\{ \tfrac{1}{2}\mathrm{K}(a_1, a_2, \ldots a_2, a_1)\,\mathrm{M} + \tfrac{1}{2}\mathrm{K}(a_1, a_2, \ldots a_2)\,\mathrm{K}(a_2 \ldots a_2) \right\}^2 \\ \quad + \mathrm{K}(a_1, a_2, \ldots a_2)\,\mathrm{M} + \mathrm{K}(a_2 \ldots a_2)^2, \end{cases}$$

where $a_1, a_2, \ldots a_n, a_n, \ldots a_2, a_1$ is the symmetric portion of the cycle of partial denominators, and the continuant notation K ( ) is explained by the example

$$\mathrm{K}(a_1, a_2, a_3, a_4) = \begin{vmatrix} a_1 & 1 & 0 & 0 \\ -1 & a_2 & 1 & 0 \\ 0 & -1 & a_3 & 1 \\ 0 & 0 & -1 & a_4 \end{vmatrix}$$

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* The asterisks are used like the upper dots in $\frac{1}{7} = \cdot 142857$.

† Proceedings Roy. Soc. Edin., Sess. 1873–4, p. 234.

In other words, reversing the order of the propositions, and denoting for shortness' sake

$$\tfrac{1}{2}\mathrm{K}(a_1, a_2, \ldots a_2, a_1)\,\mathrm{M} + \tfrac{1}{2}\mathrm{K}(a_1, a_2, \ldots a_2)\,\mathrm{K}(a_2 \ldots a_2) \text{ by R},$$

and

$$\mathrm{K}(a_1, a_2, \ldots a_2)\,\mathrm{M} + \mathrm{K}(a_2 \ldots a_2)^2 \text{ by S},$$

it is known that

$$\sqrt{(\mathrm{R}^2+\mathrm{S})} = \mathrm{R} + \underset{*}{\frac{1}{a_1}} + \frac{1}{a_2} + \ddots + \frac{1}{a_n} + \frac{1}{a_n} + \ddots + \frac{1}{a_2} + \frac{1}{a_1} + \underset{*}{\frac{1}{2\mathrm{R}}} + \ddots \qquad \ldots\ldots\ldots\ldots (a),$$

and that, on account of there being $2n+1$ terms in the cycle of partial denominators, $\mathrm{R}^2+\mathrm{S}$ must be expressible as the sum of two squares.

There is thus suggested a problem of considerable importance in the Theory of Numbers, viz., to find the two squares into which $\mathrm{R}^2+\mathrm{S}$ (*i.e.* A) may be partitioned. A solution of this has been obtained, and the object of the present paper is to place on record the rather remarkable result arrived at, and to give a verification of it.†

The expression for (A) as the sum of two squares is

$$(\mathrm{B}) \begin{cases} \Big[\tfrac{1}{2}\{\mathrm{K}(a_1 \ldots a_n)^2 - \mathrm{K}(a_1 \ldots a_{n-1})^2\}\,\mathrm{M} + \tfrac{1}{2}\mathrm{K}(a_1 \ldots a_n)\,\mathrm{K}(a_2 \ldots a_n)^3 \\ \quad -\tfrac{1}{2}\mathrm{K}(a_1 \ldots a_{n-1})\,\mathrm{K}(a_2 \ldots a_{n-1})^3 + (-1)^n \tfrac{3}{2}\mathrm{K}(a_2 \ldots a_{n-1})\,\mathrm{K}(a_2 \ldots a_n)\Big]^2 \\ + \\ \Big[\{\mathrm{K}(a_1 \ldots a_n)\,\mathrm{K}(a_1 \ldots a_{n-1})\}\,\mathrm{M} + \mathrm{K}(a_1 \ldots a_n)\,\mathrm{K}(a_2 \ldots a_{n-1})^3 \\ \quad + \mathrm{K}(a_1 \ldots a_{n-1})\,\mathrm{K}(a_2 \ldots a_n)^3\Big]^2. \end{cases}$$

For, putting

$$\mathrm{K}(a_1, \ldots, a_{n-1}) = \alpha, \quad \mathrm{K}(a_1, \ldots, a_n) = \beta,$$
$$\mathrm{K}(a_2, \ldots, a_{n-1}) = \gamma, \quad \mathrm{K}(a_2, \ldots, a_n) = \delta,$$

so that

$$\alpha\delta - \beta\gamma = (-1)^{n+1};$$

then

$$\mathrm{K}(a_1, \ldots, a_1) = \alpha^2+\beta^2,$$
$$\mathrm{K}(a_1, \ldots, a_2) = \alpha\gamma+\beta\delta,$$
$$\mathrm{K}(a_2, \ldots, a_2) = \gamma^2+\delta^2,$$

† I have chosen, for shortness' sake, to state the result and then verify it, in preference to following the process by which it was originally obtained from Lagrange's theorem on the subject. The verification here given is not, however, that at first presented to the Society, but a recast made in accordance with a suggestion of Professor Cayley.

and the identity to be established is

$$\begin{aligned}&\{\tfrac{1}{2}(\alpha^2+\beta^2)\,\mathrm{M}+\tfrac{1}{2}(\alpha\gamma+\beta\delta)(\gamma^2+\delta^2)\}^2\\&\qquad+(\alpha\gamma+\beta\delta)\,\mathrm{M}+(\gamma^2+\delta^2)^2\\&=\{-\tfrac{1}{2}(\alpha^2-\beta^2)\,\mathrm{M}+\tfrac{1}{2}\beta\delta^3-\tfrac{1}{2}\alpha\gamma^3+(-1)^n\tfrac{3}{2}\gamma\delta\}^2\\&\qquad+\{\alpha\beta\mathrm{M}+\beta\gamma^3+\alpha\delta^3\}^2.\end{aligned}$$

Making use of the relation $\alpha\delta-\beta\gamma=(-1)^{n+1}$, this may be changed into the form

$$\begin{aligned}&[\tfrac{1}{2}(\alpha^2+\beta^2)\,\mathrm{M}+\tfrac{1}{2}(\alpha\gamma^3+\beta\delta^3+\alpha\gamma\delta^2+\beta\gamma^2\delta)]^2\\&\qquad-[-\tfrac{1}{2}(\alpha^2-\beta^2)\,\mathrm{M}+\tfrac{1}{2}(-\alpha\gamma^3+\beta\delta^3-3\alpha\gamma\delta^2+3\beta\gamma^2\delta)]^2\\&=\{\alpha\beta\mathrm{M}+\beta\gamma^3+\alpha\delta^3\}^2-(\alpha\delta-\beta\gamma)^2\{(\alpha\gamma+\beta\delta)\,\mathrm{M}+(\gamma^2+\delta^2)^2\},\end{aligned}$$

and, expressing the left-hand side of this as the product of two factors and expanding the right-hand side, it becomes

$$\begin{aligned}&[\beta^2\mathrm{M}+\delta\{2\beta\gamma^2-\delta(\alpha\gamma-\beta\delta)\}][\alpha^2\mathrm{M}+\gamma\{2\alpha\delta^2+\gamma(\alpha\gamma-\beta\delta)\}]\\&=\quad\mathrm{M}^2.\alpha^2\beta^2\\&\quad+\mathrm{M}\{2\alpha\beta(\beta\gamma^3+\alpha\delta^3)-(\alpha\delta-\beta\gamma)^2(\alpha\gamma+\beta\delta)\}\\&\quad+\quad(\beta\gamma^3+\alpha\delta^3)^2\;-(\alpha\delta-\beta\gamma)^2(\gamma^2+\delta^2)^2.\end{aligned}$$

Now on the right-hand side here the term independent of M is

$$\begin{aligned}&\{\beta\gamma^3+\alpha\delta^3+(\alpha\delta-\beta\gamma)(\gamma^2+\delta^2)\}\\&\{\beta\gamma^3+\alpha\delta^3-(\alpha\delta-\beta\gamma)(\gamma^2+\delta^2)\},\end{aligned}$$

which $$\begin{aligned}&=\delta\{2\alpha\delta^2+\gamma(\alpha\gamma-\beta\delta)\}\\&\quad\gamma\{2\beta\gamma^2-\delta(\alpha\gamma-\beta\delta)\},\end{aligned}$$

as it should be; and as regards the coefficients of M, we have to show that

$$\begin{aligned}&2\alpha\beta\gamma\delta(\alpha\gamma+\beta\delta)-(\alpha^2\delta^2-\beta^2\gamma^2)(\alpha\gamma-\beta\delta)\\&\qquad=2\alpha\beta(\beta\gamma^3+\alpha\delta^3)-(\alpha\delta-\beta\gamma)^2(\alpha\gamma+\beta\delta),\end{aligned}$$

or $$(\alpha^2\delta^2+\beta^2\gamma^2)(\alpha\gamma+\beta\delta)-(\alpha^2\delta^2-\beta^2\gamma^2)(\alpha\gamma-\beta\delta)=2\alpha\beta(\beta\gamma^3+\alpha\delta^3),$$

which is the case.

Two particular cases of the general theorem thus established are of even greater interest.

When M=0, the continued fraction we have been considering becomes

$$\tfrac{1}{2}\mathrm{K}(a_1,a_2,\ldots a_2)\,\mathrm{K}(a_2\ldots a_2)$$

$$+\cfrac{1}{\underset{*}{a_1}+\cfrac{1}{a_2}+\ddots+\cfrac{1}{a_n}+\cfrac{1}{a_n}+\ddots+\cfrac{1}{a_2}+\cfrac{1}{a_1}+\cfrac{1}{\mathrm{K}(a_1\ldots a_2)\underset{*}{\mathrm{K}}(a_2\ldots a_2)}}},$$

and we learn that the integer of which it is the square root is expressible in *two* ways as the sum of two integral squares, viz.,

$$\left\{\tfrac{1}{2}\mathrm{K}(a_1 \dots a_2)\,\mathrm{K}(a_2 \dots a_2)\right\}^2 + \left\{\mathrm{K}(a_2 \dots a_2)\right\}^2,$$

and

$$\left\{\tfrac{1}{2}\mathrm{K}(a_1 \dots a_n)\,\mathrm{K}(a_2 \dots a_n)^3 - \tfrac{1}{2}\mathrm{K}(a_1 \dots a_{n-1})\,\mathrm{K}(a_2 \dots a_{n-1})^3 + (-1)^n \tfrac{3}{2}\mathrm{K}(a_2 \dots a_{n-1})\,\mathrm{K}(a_2 \dots a_n)\right\}^2$$

$$+ \left\{\mathrm{K}(a_1 \dots a_n)\,\mathrm{K}(a_2 \dots a_{n-1})^3 + \mathrm{K}(a_1 \dots a_{n-1})\,\mathrm{K}(a_2 \dots a_n)^3\right\}^2.$$

As an instance of this, we find

$$\begin{aligned}
&[16ab^4+32ab^3+8b^4+24ab^2+20b^3+18b^2+8ab+10b+2]^2 \\
&+[16ab^3+24ab^2+8b^3+16ab+12b^2+4a+8b+3\ \ \ ]^2 \\
&= [16ab^4+32ab^3+8b^4+32ab^2+20b^3+22b^2+16ab+12b+4a+3]^2 \\
&\quad +[4b^2+4b+2]^2 \\
&= \mathrm{R}^2+s^2, \text{ say,}
\end{aligned}$$

and

$$\sqrt{(\mathrm{R}^2+s^2)} = \mathrm{R} + \underset{*}{\frac{1}{2a+1}} + \frac{1}{2b+1} + \frac{1}{2b+1} + \frac{1}{2a+1} + \frac{1}{\underset{*}{2\mathrm{R}} + \ddots}$$

When $\mathrm{M} = \mathrm{K}(a_1 \dots a_2)$, the integer whose square root is equivalent to the continued fraction is expressible as the sum of two squares, and also as the sum of *three* squares. These expressions are apparent from (A) and (B), but a particular case with figures may be taken, for the purpose of giving greater definiteness to the general subject, and of exhibiting the calculus connected with it.

Let 1, 1, 1, 1, 1, 1 be the symmetric portion of the cycle of partial denominators, then

$$\begin{aligned}
\mathrm{K}(a_1 \dots a_1) &= \mathrm{K}(1,1,1,1,1,1) = 13, \\
\mathrm{K}(a_1 \dots a_2) &= \mathrm{K}(1,1,1,1,1) = 8 = \mathrm{M}, \\
\mathrm{K}(a_2 \dots a_2) &= \mathrm{K}(1,1,1,1) = 5, \\
\mathrm{K}(a_1 \dots a_n) &= \mathrm{K}(1,1,1) = 3, \\
\mathrm{K}(a_2 \dots a_n) = \mathrm{K}(a_1 \dots a_{n-1}) &= \mathrm{K}(1,1) = 2, \\
\mathrm{K}(a_2 \dots a_{n-1}) &= \mathrm{K}(1) = 1,
\end{aligned}$$

and thus, from (α), we have

$$\sqrt{5283} = 72 + \underset{*}{\frac{1}{1}} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{\underset{*}{144} + \ddots}$$

which may be readily verified,

and $5283 = 28^2+67^2$, from (B),

and $= 72^2+8^2+5^2$, from (A).

I hope to be able to communicate to the Society at some future time the solution of the general problem which is related to Göpel's theorem (*v.* Jacobi's "*Notiz über A. Göpel*," in Crelle, xxxv., p. 314) as the general problem of this paper is related to Legendre's theorem.

April 12th, 1877.

LORD RAYLEIGH, F.R.S., President, in the Chair.

Mr. Charles Pendlebury, B.A., was elected a Member.

The following communications were made to the Society:—

"On Hesse's Ternary Operator and Applications:" by J. J. Walker.

"On the Circular Relation of Möbius;" and "On the Linear Transformation of the Integral $\int \frac{du}{\sqrt{U}}$:" Prof. Cayley.

"On some cases of Parallel Motion:" Mr. Harry Hart.

"A method of solving Partial Differential Equations, which have a general first Integral, applied to equations of the third order with two independent variables:" Prof. H. W. Lloyd Tanner. (Abstract read by Mr. Tucker.)

The following presents were received:—

"Proceedings of Physical Society of London," Vol. ii., Pt. ii., June 1876 to January 1877.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome xi., 2e sémestre, 1876, and "Table des Matières et Noms d'Auteurs;" deuxième série, Tome i., Janvier 1877 (Tome xii. de la collection).

"Crelle," 82 Band, 3es und 4es Heft, Berlin, 1877.

"Monatsbericht," Nov. and Dec. 1876.

"Annali di Matematica," serie iia, Jan. 1877, Tomo 8o, Fasc 1o.

"Atti della R. Accademia dei Lincei anno cclxxiv," 1876–77, serie terza; "Transunti," Vol. i., Fasc. 3o, Feb., Fasc. 4o, Marzo, 1877. Roma.

"Bulletin de la Société Mathématique de France," Tome v., Janvier, No. 1, Paris, 1877.

"Proceedings of the Royal Society," Vol. xxv., No. 178.

"Nieuw Archief voor Wiskunde," Deel ii., Amsterdam, 1876.

"Investigation of Corrections to Hansen's Tables of the Moon, with

Tables for their application, by Simon Newcomb," forming Part iii. of papers published by the Commission of the Transit of Venus, Washington, 1876.

"Report on the Difference of Longitude between Washington and Ogden, Utah," by J. R. Eastman, Washington, 1876.

Address delivered by J. J. Sylvester, F.R.S., at Johns Hopkins University, on Commemoration Day (Feb. 22, 1877), Baltimore: from the Author.

"Ueber den Verlauf der Abels'chen Integrale bei den Curven vierten Grades" (zweiter Aufsatz), von Felix Klein, in München (from "Mathematische Annalen," Vol. xi., pp. 293–305.)

"Treatise on Integral Calculus," by B. Williamson, 1877: from the Author.

"Smithsonian Report," 1875, Washington, 1876.

"Memorie della R. Accademia di Scienze, Lettere ed Arti in Modena," Tomo xv., Tomo xvi., 1875.

"Transactions of the Cambridge Philosophical Society," Vol. xi., Pts. i., ii., iii.; Vol. xii., Pts. i., ii.

On the Circular Relation of Möbius. *By* Prof. Cayley.

[*Read April 12th*, 1877.]

In representing a given imaginary or complex quantity $u, = x+iy$, by means of the point whose coordinates are x, y, we assume in the first instance that x, y are real,—but in the results this restriction may be abandoned—for instance, if the imaginary quantities u, u', c are connected by the equation $u^2 + u'^2 = c^2$; then, writing $u = x + iy$, $u' = x'+iy'$, $c = a+bi$, we have $x^2-y^2+x'^2-y'^2 = a^2-b^2$, $xy+x'y' = ab$, equations connecting the points U, U', C which serve to represent the quantities u, u', c, and which (regarding C as a fixed point) establish a correspondence between the two variable points U, U': any given value $u, = x+iy$, is represented by the point U, and corresponding hereto we have (in the present case) two points U', viz., these are the real intersections of the curves $x'^2-y'^2 = a^2-b^2-(x^2-y^2)$, $x'y' = ab-xy$, and then the coordinates x', y' of either of these give the value $x'+iy'$ of u'.

But, the two curves once arrived at, we may for other purposes be concerned with their intersections as well imaginary as real; or, still more generally, all the quantities entering into the two equations may be regarded as imaginary.

Theoretically we seem to require two imaginary roots of unity, incommensurable and convertible, viz., taking these to be i, I, then $i^2 = -1$, $\mathrm{I}^2 = -1$, $i\mathrm{I} = \mathrm{I}i$, but without any relation between i, I: thus, in what precedes, writing I instead of i, viz., u, u', $c = x + \mathrm{I}y$, $x' + \mathrm{I}y'$, $a + b\mathrm{I}$, here each of the quantities x, y, x', y', a, b can be *ab initio* an imaginary quantity of the form $\lambda + \mu i$. But, conforming to the ordinary practice, I use i only, writing for instance $u = x + iy$, without any express statement that x, y are real; on the understanding that any equation containing such quantities, and therefore ultimately of the form $\mathrm{P} + i\mathrm{Q} = 0$, denotes the two equations $\mathrm{P} = 0$, $\mathrm{Q} = 0$ (or, what is the same thing, that we have not only the original equation, but, in addition to it, the like equation with each such quantity $x + iy$ replaced by the conjugate quantity $x - iy$): and the further understanding that, in the pair of equations, each of the quantities x, y, &c. entering therein may itself be considered as an imaginary quantity of the form $\lambda + \mu i$.

The foregoing explanation is required, for otherwise it would appear as if the circular relation of Möbius about to be explained was of necessity a relation between real points: I hold that this is *not* the case. But in all that follows I do, in fact, consider primarily the case of real points; and indeed the occasion does not arise for any explicit considera- of the case of imaginary points.

The circular relation is as follows: if in the first instance we have four points U, A, B, C on a line, and u, a, b, c their distances from any fixed point on that line; and again, U′, A′, B′, C′ four other points on a line (the same or a different line), and u', a', b', c' their distances from any fixed point on that line; then the same equation between u, a, b, c, u', a', b', c' which expresses the homographic relation between the two ranges of points U, A, B, C and U′, A′, B′, C′, expresses, when differently interpreted, the circular relation between the four points U, A, B, C in a plane, and the other four points U′, A′, B′, C′ in the same or a different plane—viz., for the new interpretation, u is used as denoting $x + iy$, the linear function of the coordinates x, y of the point U, and the like as regards the remaining quantities a, b, c, u', a', b', c'.

As in the homographic theory (but of course without the condition of being in a line), we have A, A′; B, B′; C, C′ given pairs of corresponding points: the equation now represents two equations; and these, when either of the points U, U′ is given, determine the corresponding point U′ or U.

The homographic relation may be written in the forms

$$\begin{vmatrix} 1, & u, & u', & uu' \\ 1, & a, & a', & aa' \\ 1, & b, & b', & bb' \\ 1, & c, & c', & cc' \end{vmatrix} = 0,$$

$$u-a\,.\,b-c : u-b\,.\,c-a : u-c\,.\,a-b$$
$$= u'-a'\,.\,b'-c' : u'-b'\,.\,c'-a' : u'-c'\,.\,a'-b',$$

viz., these are forms of one and the same equation: and it may be added that, if ω in the first system corresponds to ∞ in the second system, and ω' in the second system to ∞ in the first system (of course ω, ω' are not corresponding values in the two systems respectively); so that

$$\begin{vmatrix} & 1, & \omega \\ 1, & a, & a', \ aa' \\ 1, & b, & b', \ bb' \\ 1, & c, & c', \ cc' \end{vmatrix} = 0, \qquad \begin{vmatrix} 1, & & \omega \\ 1, & a, & a', \ a\alpha' \\ 1, & b, & b', \ b\beta' \\ 1, & c, & c', \ c\gamma' \end{vmatrix} = 0,$$

$$\omega-a\,.\,b-c : \omega-b\,.\,c-a : \omega-c\,.\,a-b = b'-c' : c'-a' : a'-b';$$

$$b-c : \qquad c-a : \qquad a-b$$
$$= \omega'-a'\,.\,b'-c' : \omega'-b'\,.\,c'-a' : \omega'-c'\,.\,a'-b';$$

whence also, if

$$\Lambda = \begin{vmatrix} 1, & 1, & 1 \\ a, & b, & c \\ a', & b', & c' \end{vmatrix}, = bc'-b'c+ca'-c'a+ab'-a'b,$$

then

$$-\Lambda\,.\,\omega-a = b-c\,.\,c'-a'\,.\,a'-b', \quad \Lambda\,.\,\omega'-a' = b'-c'\,.\,c-a\,.\,a-b,$$
$$-\Lambda\,.\,\omega-b = c-a\,.\,a'-b'\,.\,b'-c', \quad \Lambda\,.\,\omega'-b' = c'-a'\,.\,a-b\,.\,b-c,$$
$$-\Lambda\,.\,\omega-c = a-b\,.\,b'-c'\,.\,c-a', \quad \Lambda\,.\,\omega'-c' = a'-b'\,.\,b-c\,.\,a-b.$$

then, ω, ω' being thus determined, we have

$$\omega-a\,.\,\omega'-a' = \omega-b\,.\,\omega'-b' = \omega-c\,.\,\omega'-c' = \omega-u\,.\,\omega'-u'$$
$$= -\frac{b-c\,.\,c-a\,.\,a-b\,.\,b'-c'\,.\,c'-a'\,.\,a'-b'}{\Lambda^2}\ (=\Delta \text{ suppose}),$$

viz., we have $\omega-u\,.\,\omega'-u' =$ given value; which is the most simple form of the relation between u, u'.

Interpreting everything in the first instance in regard to the homographic ranges, the equations show that there is in the first range a point O, and in the second range a point O′, such that OA, &c. denoting distances, we have

$$OA\,.\,O'A' = OB\,.\,O'B' = OC\,.\,O'C' = OU\,.\,O'U'\ (=\Delta),$$

or, what is the same thing, if in the line of the first range we construct A_1, B_1, C_1, U_1 by the formulæ

$$OA\,.\,OA_1 = OB\,.\,OB_1 = OC\,.\,OC_1 = OU\,.\,OU_1 = \Delta,$$

that is, invert the first range in regard to the centre O and squared radius Δ, then we have a range O, A_1, B_1, C_1, U_1 *equal* to the range O′, A′, B′, C′, U′ (viz., the distances of corresponding points are equal in the two cases: or say a range O, A_1, B_1, C_1, U_1 imposable upon O′, A′, B′, C′, U′.

The like result holds for the circular relation, but the interpretation must be explained more in detail. And, first, it is to be remarked that O in the first figure is the point corresponding to any point whatever at infinity in the second figure; viz., writing $w' = \xi' + i\eta', = \infty$, then, whatever value we give to the ratio of the two infinite quantities ξ', η', we obtain the same complex value of ω, that is, the same coordinates for the point O. And, similarly, O′ in the second figure is the point corresponding to any point whatever at infinity in the first figure.

To determine O, we have the equation

$$\frac{\omega - a}{\omega - b} = \frac{b' - c'}{c' - a'} \cdot \frac{c - a}{a - b},$$

and any such equation gives at once the geometrical construction, viz., $\omega - a = \mathrm{OA}e^{i\mathrm{OA}x}$, where OA is the distance of the points O, A regarded as positive, and OAx is the inclination of the line OA regarded as drawn from A to O to the line Ax, such angle being measured in the sense Ax to Ay; where Ax, Ay are the lines drawn from A in the senses x positive and y positive respectively: and so in other cases.

The equation is therefore equivalent to the two equations

$$\frac{\mathrm{OA}}{\mathrm{OB}} = \frac{\mathrm{B'C'}}{\mathrm{C'A'}} \cdot \frac{\mathrm{CA}}{\mathrm{AB}},$$

and $$\angle \mathrm{OA}x - \angle \mathrm{OB}x = \angle \mathrm{B'C'}x - \angle \mathrm{C'A'}x + \angle \mathrm{CA}x - \angle \mathrm{AB}x.$$

The former of these expresses that O is in a certain circle having its centre on the line AB and which cuts AB and AB produced in the one or the other sense; the latter that it is in the segment described on a determinate side of AB and containing a given angle: hence O, as the intersection of the segment with the first-mentioned circle, is a uniquely determined point. Similarly O′ is a uniquely determined point.

It is not obvious how to construct Λ, from its original value as given above (but, ω being known, we can without difficulty construct it from the value $-\Lambda \,.\, \omega - a = b - c \,.\, c' - a' \,.\, a' - b'$), nor consequently Δ from its expression in terms of Λ: but, ω, ω' being known, we can construct Δ from the expression $\omega - a \,.\, \omega' - a' = \Delta$; supposing it thus constructed, $= ke^{i\theta}$ suppose, then if, with centre O and squared radius k, we invert the first figure, thereby obtaining the points A_1, B_1, C_1, U_1 such that

$$\mathrm{OA} \,.\, \mathrm{OA_1} = \mathrm{OB} \,.\, \mathrm{OB_1} = \mathrm{OC} \,.\, \mathrm{OC_1} = \mathrm{OU} \,.\, \mathrm{OU_1} = k,$$

(the points A_1, B_1, C_1, U_1 being on the lines OA, OB, OC, OU respectively.) then the equations

$$\omega - a \,.\, \omega' - a' = \omega - b \,.\, \omega' - b' = \omega - c \,.\, \omega' - c' = \omega - u \,.\, \omega' - u' = ke^{i\theta}$$

give $$\omega - a \,.\, \omega' - a' = \mathrm{OA} \,.\, \mathrm{OA}_1 e^{i\theta},$$

that is, $$\mathrm{OA} \,.\, \mathrm{O'A'} = \mathrm{OA} \,.\, \mathrm{OA}_1, \quad \text{or, simply,} \quad \mathrm{O'A'} = \mathrm{OA}_1,$$

and $$\angle \mathrm{AO}x + \angle \mathrm{A'O'}x' = \theta,$$

or, what is the same thing,

$$\angle \mathrm{A}_1\mathrm{O}x + \angle \mathrm{A'O}x' = \theta,$$

and so for the other letters, viz., we have

$$\mathrm{O'A'},\ \mathrm{O'B'},\ \mathrm{O'C'},\ \mathrm{O'U'} = \mathrm{OA}_1,\ \mathrm{OB}_1,\ \mathrm{OC}_1,\ \mathrm{OU}_1$$

respectively; and further

$$\angle\text{'s } \mathrm{A}_1\mathrm{O}x, \mathrm{B}_1\mathrm{O}x, \mathrm{C}_1\mathrm{O}x, \mathrm{U}_1\mathrm{O}x = \theta - \mathrm{A'O'}x',\ \theta - \mathrm{B'O'}x',\ \theta - \mathrm{C'O'}x',\ \theta - \mathrm{U'O'}x'$$

respectively; viz., the system of points O, A_1, B_1, C_1, U_1 is *equal* to the system O′, A′, B′, C′, U′ (that is, the distances of corresponding points and magnitudes of corresponding angles are severally equal)—but the angles A_1Ox and $A'O'x'$, &c. are in opposite senses, as appears by the just mentioned equations $A_1Ox = \theta - A'O'x'$, &c.; that is, the two figures are symmetrically equal: but the one of them is not (except by a turning over of its plane) imposable upon the other.

The conclusion is, the two figures A, B, C, U and A′, B′, C′, U′ are each of them equal by symmetry, but not super-imposably, to a figure which is the inverse of the other of them; viz., there exists in the first figure a point O, and in the second figure a point O′, such that, inverting say the first figure, with centre O and a squared radius of determinate magnitude, we obtain the points A_1, B_1, C_1, U_1, forming with O a figure equal by symmetry, but not superimposably, to the second figure A′, B′, C′, U′, O′. Hence also to any line in the first figure corresponds in the second figure a circle through O′, and to any line in the second figure there corresponds in the first figure a circle through O; or, more generally, to any circle in either figure there corresponds a circle in the other figure.

There is a particular case of peculiar interest, viz., writing for greater convenience d, d' as corresponding values in place of u, u', the system a, b, c, d corresponds homographically to itself in three different ways; that is, we may have

$$(a', b', c', d') = (b, a, d, c),\ (c, d, a, b) \text{ or } (d, c, b, a).$$

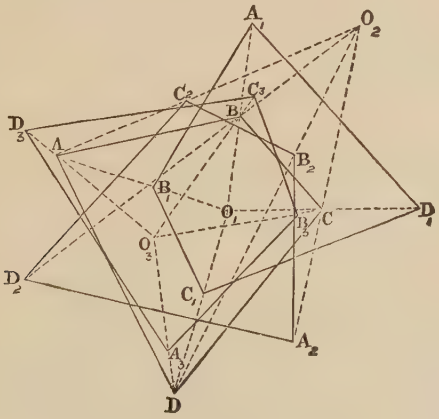
To fix the ideas, attending to the first case, we have thus the range of points (A, B, C, D) corresponding homographically to (B, A, D, C), viz., here $\omega' = \omega$, and $\omega - a \,.\, \omega - b = \omega - c \,.\, \omega - d$, that is, the corres-

ponding points U, U′ belong to the involution where A and B and also C and D are corresponding points. The like theory applies to the circular transformation: viz., the points (A, B, C, D) may correspond to (B, A, D, C), viz., there exists a point O (or say O_1) and squared radius k_1, such that, inverting the figure and marking the inverse points of A, B, C, D as B_1, A_1, D_1, C_1 respectively, the new figure O_1, A_1, B_1, C_1, D_1 is equal by symmetry, but not superimposably, to the original figure O A B C D. The equation $\omega_1 - a \,.\, \omega_1 - b = \omega_1 - c \,.\, \omega_1 - d$ gives the geometrical definition of the point O_1, viz.. this is a point such that $O_1A \,.\, O_1B = O_1C \,.\, O_1D$, and further that AB and CD subtend at O_1 equal angles: we have $\omega_1 = \dfrac{ab - cd}{a + b - c - d}$, giving for $\omega_1 - a$, $\omega_1 - b$, $\omega_1 - c$, $\omega_1 - d$ convenient expressions the first of which is $\omega_1 - a = \dfrac{c - a \,.\, d - a}{c + d - a - b}$. We hence obtain a convenient construction for O, viz., taking M for the middle point of AB and N for the middle point of CD, and drawing from A in the sense M to N a line AP, =2MN, then this equation may be written $\omega_1 - a = \dfrac{c - a \,.\, d - a}{p - a}$ (p the function $x + iy$ which belongs to the point P), and thence $O_1A = \dfrac{CA \,.\, DA}{PA}$ and

$$\angle O_1Ax = \angle CAx + \angle DAx - \angle PAx,$$

conditions which determine uniquely the position of O_1.

We may have (A, B, C, D) corresponding to (C, D, A, B) and (D, C, B, A), and for which the inversions depend on the points O_2 and O_3 respectively: I annex a figure showing the three inversions of the same four points A, B, C, D.



On the Linear Transformation of the Integral $\int \frac{du}{\sqrt{U}}$.

By Prof. Cayley, F.R.S.

[*Read April 12th*, 1877.]

The quartic function U is taken to be $= \epsilon . u-a . u-b . u-c . u-d$, where a, b, c, d are imaginary values represented in the usual manner by means of the points A, B, C, D; viz., if $a = \alpha_0 + \alpha_1 i$, then A is the point whose rectangular coordinates are α_0, α_1; and the like as regards B, C, D. And I consider chiefly the definite integrals such as $\int_a^b \frac{du}{\sqrt{U}}$, where the path is taken to be the right line from A to B. There is here nothing to fix the sign of the radical; but if at any particular point of the path we assign to it at pleasure one of its two values, then (the radical varying continuously) this determines the value at every other point of the path; and the integral defined as above is completely determinate except as to its sign, which might be fixed as above, but which is better left indeterminate. The integral, thus determinate except as to its sign, is denoted by (AB).

I wish to establish the theorem that, if the points A, B, C, D taken in this order form a convex quadrilateral, then

$$(AB) = \pm(CD), \quad (AD) = \pm(BC), \quad \text{but not} \quad (AC) = \pm(BD);$$

whereas, if the four points form a triangle and interior point, then the three equations all hold good. I regard the theorem as the precise statement of Bouquet and Briot's theorem, $A-B+C-D = 0$, or say $(OA)-(OB)+(OC)-(OD) = 0$, where the four terms are the rectilinear integrals taken from a point O to the four points A, B, C, D respectively. The two cases may be called, for shortness, the convex and the reentrant cases respectively.

To prove in the case of a convex quadrilateral that (AC) is not $= \pm(BD)$, it is sufficient to consider the integral $\int \frac{du}{\sqrt{u^4-1}}$, where A, B, C, D are the points $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$ respectively, and where, writing $v = iu$, it at once appears that we have

$$\int_{-1}^{1} \frac{du}{\sqrt{u^4-1}} = \pm i \int_{-i}^{i} \frac{du}{\sqrt{u^4-1}},$$

that is, $$(AC) = \pm i(BD), \quad \text{not} \quad (AC) = \pm(BD).$$

But I consider the general question of the linear transformation, viz., if a', b', c', d' correspond homographically to a, b, c, d, then to represent these values a', b', c', d' we have the points A′, B′, C′, D′,

connected with A, B, C, D according to the circular relation of Möbius; and then, making u', a', b', c', d' correspond homographically to u, a, b, c, d, and representing in like manner the variables u, u' by the points U, U' respectively, we have the circular relation between the two systems U, A, B, C, D and U', A', B', C', D'.

Before going further I remark that the distinction of the convex and reentrant cases is not an invariable one; the figures are transformable the one into the other. Thus, taking C on the line BD (that is, between B and D, not on the line produced), there is not this relation between B', C', D', and the figure A'B'C'D' is convex or reentrant as the case may be. Giving to C an infinitesimal displacement to the one side or the other of the line BD, we have in the one case a convex figure, in the other case a reentrant figure ABCD; but the corresponding displacement of C' being infinitesimal, the figure A'B'C'D' remains for either displacement, convex or reentrant, as it originally was; that is, we have a convex figure ABCD and a reentrant figure ABCD, each corresponding to the figure A'B'C'D' (which is convex, or else reentrant, as the case may be).

Writing for convenience

$$\begin{aligned} &\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{f},\ \mathrm{g},\ \mathrm{h} = b-c,\ c-a,\ a-b,\ a-d,\ b-d,\ c-d, \\ &\mathrm{a}',\ \mathrm{b}',\ \mathrm{c}',\ \mathrm{f}',\ \mathrm{g}',\ \mathrm{h}' = b'-c',\ c'-a',\ a'-b',\ a'-d',\ b'-d',\ c'-d', \end{aligned}$$

so that identically

$$\mathrm{af}+\mathrm{bg}+\mathrm{ch} = 0, \qquad \mathrm{a'f'}+\mathrm{b'g'}+\mathrm{c'h'} = 0,$$

then the homographic relation between (a, b, c, d), (a', b', c', d') may be written in the forms

$$\mathrm{af} : \mathrm{bg} : \mathrm{ch} = \mathrm{a'f'} : \mathrm{b'g'} : \mathrm{c'h'},$$

or, what is the same thing, there exists a quantity N such that

$$\frac{\mathrm{a'f'}}{\mathrm{af}} = \frac{\mathrm{b'g'}}{\mathrm{bg}} = \frac{\mathrm{c'h'}}{\mathrm{ch}} = \mathrm{N}^2.$$

The relation between u, u' may be written in the forms

$$\frac{u'-a'}{u'-d'} = \mathrm{P}\frac{u-a}{u-d}, \quad \frac{u'-b'}{u'-d'} = \mathrm{Q}\frac{u-b}{u-d}, \quad \frac{u'-c'}{u'-d'} = \mathrm{R}\frac{u-c}{u-d};$$

and then, writing for u, u' their corresponding values, we find

$$\mathrm{P} = \frac{\mathrm{b'h}}{\mathrm{bh'}} = \frac{\mathrm{c'g}}{\mathrm{cg'}}, \quad \mathrm{Q} = \frac{\mathrm{c'f}}{\mathrm{cf'}} = \frac{\mathrm{a'h}}{\mathrm{ah'}}, \quad \mathrm{R} = \frac{\mathrm{a'g}}{\mathrm{ag'}} = \frac{\mathrm{b'f}}{\mathrm{bf'}},$$

giving $\mathrm{f}^2\mathrm{PN}^2 = \mathrm{f}'^2\mathrm{QR}$, $\mathrm{g}^2\mathrm{QN}^2 = \mathrm{g}'^2\mathrm{RP}$, $\mathrm{h}^2\mathrm{RN}^2 = \mathrm{h}'^2\mathrm{PQ}$,

$$\sqrt{\mathrm{PQR}} = \frac{\mathrm{fgh}}{\mathrm{f'g'h}}, \mathrm{N}^3.$$

Differentiating any one of the equations in (u, u'), for instance the first of them, we find

$$\frac{f'du'}{(u'-d')^2} = \frac{fPdu}{(u-d)^2};$$

and then forming the equation

$$\frac{\sqrt{\epsilon.u'-a'.u'-b'.u'-c'.u'-d'}}{(u'-d')^2} = \pm \frac{\sqrt{PQR}\,\sqrt{\epsilon.u-a.u-b.u-c.u-d}}{(u-d)^2},$$

and attending to the relation $f^2PN^2 = f'^2QR$, we obtain

$$\pm \frac{N\,du'}{\sqrt{U'}} = \frac{du}{\sqrt{U}},$$

which is the differential relation between u, u'.

We have in connection with A, B, C, D the point O, and in connection with A′, B′, C′, D′ the point O′; and as U describes the right line AB, U′ describes the arc not containing O′ of the circle A′B′O′ (for observe that O′ corresponds in the second figure to the point at infinity on the line AB, viz., as U passes from U, not passing through the point at infinity, U′ must pass from A′ to B′, not passing through the point O′, that is, it must describe, not the arc A′O′B′, but the remaining arc $2\pi - A'O'B'$, say this is the arc $\overset{\frown}{A'B'}$). The integral in regard to u' is thus not the rectilinear integral (A′B′), but the integral along the just-mentioned circular arc, say this is denoted by $(\overset{\frown}{A'B'})$; and we thus have

$$(AB) = \pm N\,(\overset{\frown}{A'B'}).$$

But we have $(\overset{\frown}{A'B'})$ = or not = (A′B′), according as the chord A′B′ and the arc $\overset{\frown}{A'B'}$ do not include between them either of the points C′, D′, or include between them one or both of these points; and in the same cases respectively

$$(AB) = \text{or not} = \pm N\,(A'B').$$

Of course we may in any way interchange the letters, and write under the like circumstances (AC) = or not = ± N (A′C′), &c.

Suppose now that ABCD is a convex quadrilateral, and consider first in regard to (AB), and next in regard to (AC), the three transformations A′B′C′D′ = BADC, = CDAB and = DCBA respectively. We have here a figure as in the paper "On the Circular Relation of Möbius," the points O_1, O_2, O_3 belonging to the three cases respectively. It will be observed in the figure, and it is easy to see generally, that the points O_1 and O_3 are interior, the point O_2 exterior. We have N = 1, and therefore

(AB) = or not = ± (AB), = or not = ± (CD), = or not = ± (CD),

according as

(1) chord AB and arc $\overset{\smile}{AB}$ of ABO_1 do not or do inclose C and D or either of them;

(2) chord CD and arc $\overset{\smile}{CD}$ of CDO_2 do not or do inclose A and B or either of them;

(3) chord CD and arc $\overset{\smile}{CD}$ of CDO_3 do not or do inclose A and B or either of them.

The first test gives merely the identity $(AB) = \pm(AB)$; the other two each of them give $(AB) = \pm(CD)$, as is seen from the positions of the points O_1, O_2, O_3.

Next apply the test to AC; we have

$$(AC) = \text{or not} = \pm(BD), = \text{or not} = \pm(AC), = \text{or not} = \pm(BD),$$

according as

(1) chord AC and arc $\overset{\smile}{AC}$ of ACO_1 do not or do inclose B and D or either of them;

(2) chord BD and arc $\overset{\smile}{BD}$ of BDO_2 do not or do inclose A and C or either of them;

(3) chord BD and arc $\overset{\smile}{BD}$ of BDO_3 do not or do inclose A and C or either of them.

In the second case neither A or C is inclosed, but we have merely the identity $(AC) = \pm(AC)$; in the first case B is inclosed, and in the third case C is inclosed; and the tests each give (AC) not $= \pm(BD)$.

I have not taken the trouble of drawing the figure for a reentrant ABCD; the mere symmetry is here enough to show that, having one, we have all three of the relations in question,

$$(AD) = \pm(BC), \quad (BD) = \pm(CA), \quad (CD) = \pm(AB).$$

A method of solving Partial Differential Equations which have a general first integral, applied to Equations of the Third Order with two independent variables. *By* H. W. Lloyd Tanner, M.A.

[*Read April 12th*, 1877.]

The method employed in the present paper is identical with that applied by the author to the solution of equations of the second order with any number of variables, in a paper published in the Society's Proceedings.* This method is applicable to equations of any order and with any number of variables (Art. 8), but it is here employed in the discussion of equations of the third order with two independent variables only.

* Vol. vii., p. 75.

In the first part of the paper it is proved that all equations of the third order which have a general first integral are included in the form

$$\mathrm{H}(a\gamma-\beta^2)+\mathrm{K}(a\delta-\beta\gamma)+\mathrm{L}(\beta\delta-\gamma^2)+\mathrm{A}a-\mathrm{B}\beta+\mathrm{C}\gamma-\mathrm{D}\delta+\mathrm{M}=0 \quad \ldots\ldots\ldots\ldots \quad (2),$$

where the Greek letters represent the differential coefficients of z of the third order, and the Roman letters are functions of the other differential coefficients of z, together with z, x, y.

The first integrals of such an equation are determined by a system of ten linear homogeneous equations of the first order, which are equivalent to three, and only three independent equations. The coefficients of this auxiliary system are of two kinds. Those of the first kind are identical with some of the coefficients of (2). Those of the second group are one-, two-, or three-valued functions of the coefficients of (2); and corresponding to these we have one, two, or three auxiliary systems, each of which may serve to determine a first integral.

The rest of the paper is devoted to a detailed discussion of three classes of equations of the third order, the classes being defined thus :—

Class 1. $\mathrm{K}^2-\mathrm{HL}$ does not vanish.

„ 2. $\mathrm{K}^2-\mathrm{HL}=0$; but H, K, L do not all vanish.

„ 3. $\mathrm{H}=\mathrm{K}=\mathrm{L}=0$; or equations linear in α, β, γ, δ.

Equations of the first class have, at most, one general first integral. In class 2 two general first integrals (which may be identical) may exist; and if so we can find a second integral, *i.e.*, an integral of the first order. Of linear equations we may be able to find three first integrals, two or all of which may be identical. In this case it appears possible to determine also a third or final integral, *i.e.*, an equation between z, x, y involving three arbitrary functions and satisfying the given equation.

Two points of some interest in the theory of equations of the second order incidentally occur. First, we succeed in integrating some equations of the second order which involve an arbitrary function in their expression. Secondly, this integration is effected by the help of two other equations of the second order; viz., here we have a method of integration analogous to that employed in the case of equations of the first order.

1. *Abbreviations.* The following usual abbreviations are employed throughout the paper :—

p, q for $\dfrac{dz}{dx}$, $\dfrac{dz}{dy}$,

r, s, t „ $\dfrac{d^2z}{dx^2}$, $\dfrac{d^2z}{dx\,dy}$, $\dfrac{d^2z}{dy^2}$,

α, β, γ, δ „ $\dfrac{d^3z}{dx^3}$, $\dfrac{d^3z}{dx^2\,dy}$, $\dfrac{d^3z}{dx\,dy^2}$, $\dfrac{d^3z}{dy^3}$.

If u be a function of x, y, z, and its differential coefficients, we have three distinct symbols for the differential coefficients of u with respect to x. In $\frac{du}{dx}$ all quantities in u except x are regarded as constant. In $\frac{d\,.\,u}{dx}$, z and all its differential coefficients are regarded as functions of x. The third symbol is $\left(\frac{du}{dx}\right)$; and in this, supposing we are dealing with equations of the n^{th} order, we regard z and its differential coefficients, up to the $\overline{n-2}^{\text{th}}$ order inclusive, as functions of x; the other differential coefficients of z being considered as constant. For example, suppose we are dealing with equations of the third order, and that u is a function of r, p, q, z, x; then we have

$$\frac{d\,.\,u}{dx} \equiv \frac{du}{dr}\,a + \left(\frac{du}{dx}\right)$$
$$\equiv \frac{du}{dr}\,a + \frac{du}{dp}\,r + \frac{du}{dq}\,s + \frac{du}{dz}\,p + \frac{du}{dx}.$$

The notation for Jacobian determinants proposed by the late Prof. Donkin will be used. When either of the differentiations is with respect to one of the independent variables, the bracketed differential coefficients are implied. Thus we write

$$\frac{d\,(u_1, u_2)}{d\,(r, s)} \text{ for } \frac{du_1}{dr}\,.\,\frac{du_2}{ds} - \frac{du_1}{ds}\,.\,\frac{du_2}{dr},$$
$$\frac{d\,(u_1, u_2)}{d\,(r, x)} \text{ for } \frac{du_1}{dr}\,.\left(\frac{du_2}{dx}\right) - \left(\frac{du_1}{dx}\right).\,\frac{du_2}{dr}.$$

2. *Genesis of the Equation.*—Let u_1, u_2 be two mutually independent functions of x, y, z, p, q, r, s, t; and F a symbol of an arbitrary function. Then we seek the form of the equations of the third order which have general first integrals of the form

$$\mathrm{F}\,(u_1, u_2) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

We assume that F has two arguments u_1, u_2; because, if there were only one argument, (1) would satisfy *two* independent equations of the third order, and therefore could not be a general first integral of either. On the other hand, were there more than two independent arguments, (1) would not satisfy any equation of the third order independently of the form of F.

Differentiate (1) with respect to x, y in turn, and from the resulting equations eliminate the arbitrary ratio $\frac{d\mathrm{F}}{du_1}$, $\frac{d\mathrm{F}}{du_2}$. The result is

$$\begin{vmatrix} \frac{d\,.\,u_1}{dx}, & \frac{d\,.\,u_2}{dx} \\ \frac{d\,.\,u_1}{dy}, & \frac{d\,.\,u_2}{dy} \end{vmatrix} = 0.$$

Replacing $\frac{d \cdot u_1}{dx}$, &c. by their values; viz.

$$\frac{d \cdot u_1}{dx} = \frac{du_1}{dr}\alpha + \frac{du_1}{ds}\beta + \frac{du_1}{dt}\gamma + \left(\frac{du_1}{dx}\right),$$
$$\&c. = \&c.;$$

and, expanding the determinant, we get the required form of the equation of the third order; which is

$$\mathrm{H}\begin{vmatrix}\alpha, & \beta\\ \beta, & \gamma\end{vmatrix} + \mathrm{K}\begin{vmatrix}\alpha, & \beta\\ \gamma, & \delta\end{vmatrix} + \mathrm{L}\begin{vmatrix}\beta, & \gamma\\ \gamma, & \delta\end{vmatrix} + \mathrm{A}\alpha - \mathrm{B}\beta + \mathrm{C}\gamma - \mathrm{D}\delta + \mathrm{M} = 0 \ldots\ldots\ldots (2),$$

where

$$\left.\begin{aligned}
&\mathrm{H} = \mu\frac{d(u_1, u_2)}{d(r, s)};\quad \mathrm{K} = \mu\frac{d(u_1, u_2)}{d(r, t)};\quad \mathrm{L} = \mu\frac{d(u_1, u_2)}{d(s, t)};\\
&\mathrm{A} = \mu\frac{d(u_1, u_2)}{d(r, y)};\quad \mathrm{D} = \mu\frac{d(u_1, u_2)}{d(t, x)};\quad \mathrm{M} = \mu\frac{d(u_1, u_2)}{d(x, y)};\\
&\mathrm{B} = \mu\left\{\frac{d(u_1, u_2)}{d(r, x)} + \frac{d(u_1, u_2)}{d(y, s)}\right\};\\
&\mathrm{C} = \mu\left\{\frac{d(u_1, u_2)}{d(t, y)} + \frac{d(u_1, u_2)}{d(x, s)}\right\};
\end{aligned}\right\} \ldots\ldots (3),$$

μ being an indeterminate multiplier.

Hence it appears that all equations of the third order which have a general first integral (1) must be of the form (2). The converse, that all equations of the form (2) have a general first integral, is not true. In fact, the equations (3) show that the eight coefficients H, K, L, A, B, C, D, M are expressible in terms of three independent quantities μ, u_1, u_2; so that these coefficients are subject to five equations of condition. Now, if from (3) we could eliminate μ, u_1, u_2, we should obtain these five conditions, and the three remaining equations would, after the elimination of μ, give the values of u_1, u_2; so that the problem would be completely solved. But this direct process appears to be impracticable; at all events, in the corresponding problem for equations of the second order, Boole only partially succeeded ("Diff. Equations," Supp. Vol., pp. 148—150). In this paper an indirect process will be employed, which will give the values of u_1, u_2 when the coefficients H, K, &c. satisfy the five conditions mentioned above; and will, implicitly, give these conditions.

3. *Equations for* u_1, u_2.—Consider the system of equations

$$\begin{Vmatrix}\left(\frac{du}{dx}\right), & \left(\frac{du}{dy}\right), & \frac{du}{dr}, & \frac{du}{ds}, & \frac{du}{dt}\\ \left(\frac{du_1}{dx}\right), & \left(\frac{du_1}{dy}\right), & \frac{du_1}{dr}, & \frac{du_1}{ds}, & \frac{du_1}{dt}\\ \left(\frac{du_2}{dx}\right), & \left(\frac{du_2}{dy}\right), & \frac{du_2}{dr}, & \frac{du_2}{ds}, & \frac{du_2}{dt}\end{Vmatrix} = 0 \ldots\ldots\ldots (4).$$

All these equations are satisfied when $u = u_1$ or when $u = u_2$; so that u_1, u_2 are included in the common solutions of (4). Also any two independent solutions of (4) may be put for u_1, u_2 in (1). For the equation

$$\frac{d(u, u_1, u_2)}{d(r, s, t)} = 0$$

means that u is a function of u_1, u_2 so far as r, s, t are concerned. Hence the system (4) means that u is a function of u_1, u_2 as far as x, y, z, p, q, r, s, t are concerned, viz., it is a function of u_1, u_2 and constants only. On putting any two independent values of u for u_1, u_2, in (1), this equation becomes

$$\mathrm{F}\{f_1(u_1, u_2),\ f_2(u_1, u_2)\} = 0,$$

which is neither more nor less general than

$$\mathrm{F}\{u_1, u_2\} = 0.$$

4. The ten equations (4) may be written in a more convenient form, by means of the equations (3) and the following:—

$$\left.\begin{aligned} b = \mu \cdot \frac{d(u_1, u_2)}{d(r, x)};\ b = \mu \cdot \frac{d(u_1, u_2)}{d(y, s)};\\ c = \mu \cdot \frac{d(u_1, u_2)}{d(t, y)};\ c' = \mu \cdot \frac{d(u_1, u_2)}{d(x, s)};\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (5),$$

which give

$$\mathrm{B} = b + b';\ \mathrm{C} = c + c' \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6).$$

On making these substitutions, (4) become

$$\left.\begin{aligned}
-\mathrm{A}\left(\frac{du}{dx}\right) + b\left(\frac{du}{dy}\right) + \mathrm{M}\frac{du}{dr} &= 0 \quad \ldots\ldots \text{(i.)}\\
b'\left(\frac{du}{dx}\right) - c'\left(\frac{du}{dy}\right) + \mathrm{M}\frac{du}{ds} &= 0 \quad \ldots\ldots \text{(ii.)}\\
-c\left(\frac{du}{dx}\right) + \mathrm{D}\left(\frac{du}{dy}\right) + \mathrm{M}\frac{du}{dt} &= 0 \quad \ldots\ldots \text{(iii.)}\\
\mathrm{H}\left(\frac{du}{dx}\right) - c'\frac{du}{dr} - b\frac{du}{ds} &= 0 \quad \ldots\ldots \text{(iv.)}\\
\mathrm{K}\left(\frac{du}{dx}\right) + \mathrm{D}\frac{du}{dr} - b\frac{du}{dt} &= 0 \quad \ldots\ldots \text{(v.)}\\
\mathrm{L}\left(\frac{du}{dx}\right) + \mathrm{D}\frac{du}{ds} + c'\frac{du}{dt} &= 0 \quad \ldots\ldots \text{(vi.)}\\
\mathrm{H}\left(\frac{du}{dy}\right) - b'\frac{du}{dr} - \mathrm{A}\frac{du}{ds} &= 0 \quad \ldots\ldots \text{(vii.)}\\
\mathrm{K}\left(\frac{du}{dy}\right) + c\frac{du}{dr} - \mathrm{A}\frac{du}{dt} &= 0 \quad \ldots\ldots \text{(viii.)}\\
\mathrm{L}\left(\frac{du}{dy}\right) + c\frac{du}{ds} + b'\frac{du}{dt} &= 0 \quad \ldots\ldots \text{(ix.)}\\
\mathrm{L}\frac{du}{dr} - \mathrm{K}\frac{du}{ds} + \mathrm{H}\frac{du}{dt} &= 0 \quad \ldots\ldots \text{(x.)}
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (7).$$

Any of these equations may be written down by the help of (3) and (6); that containing $\frac{du}{dr}$, $\frac{du}{ds}$, $\frac{du}{dt}$, for example, being

$$\frac{du}{dr} \cdot \frac{d(u_1, u_2)}{d(s, t)} + \frac{du}{ds} \cdot \frac{d(u_1, u_2)}{d(t, r)} + \frac{du}{dt} \cdot \frac{d(u_1, u_2)}{d(r, s)} = 0,$$

which becomes (x.) of (7). The signs of the various terms are most easily obtained by preserving the same cyclical order in the denominators; and remembering that

$$\frac{d(u_1, u_2)}{d(\xi, \eta)} \equiv -\frac{d(u_1, u_2)}{d(\eta, \xi)}.$$

Suppose now that we can determine the values of b, b', c, c'. Then the equations (7) include all the first integrals of (2) amongst their common solutions:—for we only predicated of u_1, u_2 that they are such as to make (1) a general first integral of (2); and this is true of the arguments of any general first integral of (2). Also any solution common to (7) is a first integral of (1).

5. We now propose to find the relations by which b, &c. may be determined. This is most easily done by using the identity

$$\begin{vmatrix} k_1, k_2 \\ l_1, l_2 \end{vmatrix} \times \begin{vmatrix} k_3, k_4 \\ l_3, l_4 \end{vmatrix} + \begin{vmatrix} k_1, k_3 \\ l_1, l_3 \end{vmatrix} \times \begin{vmatrix} k_4, k_2 \\ l_4, l_2 \end{vmatrix} + \begin{vmatrix} k_1, k_4 \\ l_1 l_4 \end{vmatrix} \times \begin{vmatrix} k_2, k_3 \\ l_2, l_3 \end{vmatrix} = 0.$$

If in this identity we replace the k's by differential coefficients of u_1, and the l's by the corresponding differential coefficients of u_2, and reduce by means of (3) and (5), we get the equations

$$\left.\begin{aligned} bb' &= Ac' - HM \ldots\ldots\ldots \text{(i.)} \\ bc &= AD + KM \ldots\ldots\ldots \text{(ii.)} \\ cc' &= Db' - LM \ldots\ldots\ldots \text{(iii.)} \\ Lb + Kc' + HD &= 0 \ldots\ldots \text{(iv.)} \\ Kb' + Hc + LA &= 0 \ldots\ldots \text{(v.)} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (8).$$

These equations are not all independent; but it is convenient to keep them all for future reference. With (6) they serve to determine b, b', c, c' and in some cases they give one relation which must be satisfied by the coefficients H, K, &c.

6. The results of the last paragraph may be obtained in another way, and expressed in a form which will probably be found more convenient in practice. The ten equations (4) or (7) are not all independent; in fact, they are equivalent to three, and only three, mutually independent equations. To prove the first proposition, we observe that the determinants of the second order, formed from the last two rows of (4), cannot all vanish; for, if they did, by (3), all the coefficients of the original

equation would vanish. Say then that $\frac{d(u_1, u_2)}{d(x, y)}$ does not vanish. Then we can form three equations by taking each of the last three columns in turn with the first two columns. Each of the equations thus formed contains one, and only one, of the differential coefficients $\frac{du}{dr}$, $\frac{du}{ds}$, $\frac{du}{dt}$; and cannot therefore be an algebraical consequence of the other two. [This is very easily seen in (7); if M do not vanish (i.), (ii.), (iii.) cannot be mutually dependent.] But when these three independent equations are satisfied, every determinant of the matrix must vanish; so that all the other equations are satisfied.

7. Hence we get a rule for solving any given equation of the form (2). Form the ten equations (7), putting for H, K, &c. their proper values, but leaving b, b', c, c' indeterminate. Choose any three independent equations of the group, and express that the rest are algebraical consequences of these three. [We should thus, in the general case, reproduce (8).] If this be impossible the given equation has no first integral; but when a first integral exists we shall get equations which with (6) give the values of b, b', c, c'. The solution of the given equation is thus made to depend on the solution of a system of linear homogeneous partial differential equations of the first order.

Generalization of the preceding results.

8. The extension of the method to equations of higher orders and with more than two independent variables presents no difficulty. Suppose we have an equation of the n^{th} order with m independent variables, which has a general first integral of the form

$$\mathrm{F}\{u_1, u_2, \ldots u_m\} = 0,$$

F being an arbitrary function, and $u_1, \ldots u_m$ m independent functions of $x_1, x_2, \ldots z$ and its differential coefficients up to the $\overline{n-1}^{\text{th}}$ order inclusive. The equation of the n^{th} order must be of the form

$$\begin{vmatrix} \frac{d \cdot u_1}{dx_1}, & \ldots\ldots & \frac{d \cdot u_m}{dx_1} \\ \frac{d \cdot u_2}{dx_1}, & \ldots\ldots & \ldots \\ \ldots & \ldots\ldots & \ldots \\ \frac{d \cdot u_m}{dx_1}, & \ldots\ldots & \ldots \end{vmatrix} = 0,$$

and this may be developed by a process given in the paper already cited.* The result may be stated in this way:—There are m_1,

* Proceedings of the London Mathematical Society, Vol. vii., pp. 77—79.

$= \frac{\lfloor m+n-2}{\lfloor m-1 \lfloor n-1}$, differential coefficients of z of the $\overline{n-1}^{\text{th}}$ order. Consider a matrix of m rows each formed by differentiating these m_1 differential coefficients, with respect to one of the m independent variables. Then any minor determinant that can be formed from this matrix may form one factor of a term in the equation of the n^{th} order. The coefficient of this term will be a Jacobian determinant of $u_1, u_2, \ldots u_m$, containing derivatives of u with respect to r of the $\overline{n-1}^{\text{th}}$ differential coefficients of z, if r be the order of the determinant, which is its cofactor.

Corresponding to the equations (4) we form a matrix consisting of $m+1$ rows, viz.. one each for $u, u_1, \ldots u_m$. There are m columns, each containing differential coefficients of u with respect to one of the independent variables. Also there are m_1 columns corresponding to the m_1 differential coefficients of z of the $\overline{n-1}^{\text{th}}$ order. The system comprises

$$\frac{\lfloor m_1+m}{\lfloor m+1 \lfloor m_1-1}$$

equations, of which m_1, and only m_1, are independent. Of these equations it may be shown, as in paragraph 3, that they include all first integrals, and only first integrals of the given equation of the n^{th} order. The coefficients may be expressed, some directly, some indirectly, in terms of coefficients of the given equation; in the latter case by expressing that the auxiliary system of equations is equivalent to m_1 mutually independent equations. The second group of coefficients may also be obtained in a manner analogous to that of paragraph 5, by means of a generalization of the identity there used.*

When, then, an equation of the n^{th} order with m independent variables admits of a general first integral, the discovery of that integral depends on the solution of

$$\frac{\lfloor m+n-2}{\lfloor m-1 \lfloor n-1}$$

linear homogeneous equations of the first order; and, as will appear in the sequel, the first integrals of the given equation may suffice to give the subsequent integrals.

Discussion of Equation of Third Order—continued.

9. In preceding paragraphs it has been shown that the first integrals of

$$\mathrm{H}\begin{vmatrix} \alpha, & \beta \\ \beta, & \gamma \end{vmatrix} + \mathrm{K}\begin{vmatrix} \alpha, & \gamma \\ \beta, & \delta \end{vmatrix} + \mathrm{L}\begin{vmatrix} \beta, & \gamma \\ \gamma, & \delta \end{vmatrix}$$
$$+ \mathrm{A}\alpha - \mathrm{B}\beta + \mathrm{C}\gamma - \mathrm{D}\delta + \mathrm{M} = 0 \quad \ldots\ldots\ldots\ldots (2)$$

* "Educational Times," Question 5247.

are the common solutions of (7). To obtain these solutions the well-known method proposed by Boole ("Differential Equations," Supplementary Vol., Chap. xxv.) may be employed. This method consists in forming from the given system of equations other equations which are satisfied by all their common solutions. When all possible independent equations have been formed, the common solutions are given by the integration of exact differentials.

It is convenient, before applying this method, to separate the general equation (2) into the following groups:—

Class I. K^2-HL does not vanish.
(1) K does not vanish;
(2) K does vanish.

„ II. $K^2-HL=0$; but H, K, L do not all vanish.

„ III. H, K, L all vanish.
(1) A, D do not both vanish;
(2) $A=D=0$.

The reasons for this classification will appear in the sequel.

Class I. K^2-HL *different from zero.*

10. *Determination of b, &c.*—From (iv.), (v.) of (8), and (6), we get, by eliminating b', c, the equations

$$\left.\begin{aligned} Lb+Kc'+HD&=0\\ Kb+Hc'-KB-HC-LA&=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

Hence
$$\left.\begin{aligned} (K^2-HL)\,b&=\ AKL+BK^2+CHK+DH^2\\ (K^2-HL)\,c'&=-AL^2-BKL-CHL-DHK\end{aligned}\right\}$$
Similarly,
$$\left.\begin{aligned} (K^2-HL)\,c&=\ AL^2+BKL+CK^2+DHK\\ (K^2-HL)\,b'&=-AKL-BHL-CHK-DH^2\end{aligned}\right\}\quad\ldots\ldots(10).$$

In this case, then, we have only one auxiliary set of equations for determining a first integral; for b, b', c, c' are determined without ambiguity.

Substituting these values in (ii.) of (8), we get an equation of condition, viz.,

$$(AKL+BK^2+CHK+DH^2)\,(AL^2+BKL+CK^2+DHK)$$
$$=(K^2-HL)^3\,(KM+AD)\ldots\ldots(11).$$

(1) K *different from zero.*

11. Assuming in the first place that K does not vanish, the three independent equations (v.), (viii.), (x.) of (7) may be used to find the first integral. It is convenient also to take K to be unity; this, being equivalent to dividing the equation throughout by K, does not affect the generality of the results. The auxiliary system then becomes

$$\left.\begin{aligned}
\Delta_1 u &\equiv \frac{du}{dx} + p\frac{du}{dz} + r\frac{du}{dp} + s\frac{du}{dq} + \mathrm{D}\frac{du}{dr} - b\frac{du}{dt} = 0 \\
\Delta_2 u &\equiv \frac{du}{dy} + q\frac{du}{dz} + s\frac{du}{dp} + t\frac{du}{dq} + c\frac{du}{dr} - \mathrm{A}\frac{du}{dt} = 0 \\
\Delta_3 u &\equiv \frac{du}{ds} \qquad\qquad - \mathrm{L}\frac{du}{dr} - \mathrm{H}\frac{du}{dt} = 0
\end{aligned}\right\} \ldots\ldots\ldots(12).$$

Now Boole has shown that the common solutions of (12) also satisfy the linear homogeneous equations of the first order,

$$(\Delta_i\Delta_j - \Delta_j\Delta_i)\, u = 0, \quad (i, j = 1, 2, 3,)$$

where the Δ's are as defined in (12). It is easy to write down these equations, for "the form of the result is the same as if the Δ from either equation operated only on the coefficients of the other." In the present instance the equations are

$$\left.\begin{aligned}
&\frac{du}{dp} - \mathrm{H}\frac{du}{dq} + (\Delta_3 c + \Delta_2 \mathrm{L})\frac{du}{dr} - (\Delta_3 \mathrm{A} - \Delta_2 \mathrm{H})\frac{du}{dt} = 0 \\
\mathrm{L}&\frac{du}{dp} - \frac{du}{dq} - (\Delta_1 \mathrm{L} + \Delta_3 \mathrm{D})\frac{du}{dr} - (\Delta_1 \mathrm{H} - \Delta_3 b)\frac{du}{dt} = 0 \\
c&\frac{du}{dp} + b\frac{du}{dq} + (\Delta_2 \mathrm{D} - \Delta_1 c)\frac{du}{dr} - (\Delta_2 b - \Delta_1 \mathrm{A})\frac{du}{dt} = 0
\end{aligned}\right\} \ldots\ldots(13).$$

To these may be added another; for the common solutions of the first of (12) and the first of (13) satisfy an equation of the form

$$\frac{du}{dz} + \mathrm{P}\frac{du}{dp} + \&\mathrm{c}. = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (14).$$

It is easily seen that neither of the equations (13) can be an algebraical consequence of (12); for neither of them contains $\frac{du}{dx}$, $\frac{du}{dy}$, or $\frac{du}{ds}$. For the same reason (14) cannot be a consequence of (12); and it cannot be a consequence of (13) because the latter do not involve $\frac{du}{dz}$.

12. From the last paragraph it appears that, when the three equations (13) are mutually independent, the common solutions of (12) satisfy seven independent equations. But these equations have eight independent variables, x, y, z, p, q, r, s, t. Hence there cannot be more than one common solution of (12); or, in other words, we may be able in this case to obtain a first integral of (2) involving one arbitrary constant; but there cannot be a first integral of the form (1) involving an arbitrary function.

13. Consider now the case in which (13) are not mutually independent. It is clear that the first two equations must be independent, since HL is by hypothesis not equal to 1 (*i. e.*, K^2). Hence the only

case to be discussed is that in which the third equation of (13) is an identity or an algebraical consequence of the first two. Should this be the case, the system (12), (13), (14) reduce to six independent equations. If all the equations derived from these be algebraical consequences of them, there are two common solutions of (12), and we can find a first integral of the proposed form. The integration cannot, however, be carried any further.

Some examples are appended.

14. *Example* 1. $\alpha\delta - \beta\gamma = 0$ ..... (i.)

Here all the coefficients vanish except K. The equations (13) become

$$\frac{du}{dp} = 0, \quad \frac{du}{dq} = 0 \quad \text{.....(ii.), (iii.)}$$

By means of these, (12) reduce to

$$\frac{du}{dx} + p\frac{du}{dz} = 0, \quad \frac{du}{dy} + q\frac{du}{dz} = 0 \text{.....(iv.), (v.),}$$

$$\frac{du}{ds} = 0 \text{ .....(vi.)}$$

The common solutions of (iv.) and (i.), or of (v.) and (ii.), satisfy

$$\frac{du}{dz} = 0.$$

Hence the ordinary differential equations giving the common solutions of (iv.), (v.), (vi.) are

$$dr = 0, \quad dt = 0,$$

giving $u_1 = r, \quad u_2 = t.$

The first integral of (i.) is therefore

$$\mathrm{F}(r, t) = 0.$$

15. *Example* 2.

$$t(\alpha\gamma - \beta^2) - s(\alpha\delta - \beta\gamma) + r(\beta\delta - \gamma^2) = 0 \text{ .....(i.).}$$

Here $\mathrm{K} = 1, \ \mathrm{H} = -\frac{t}{s}, \ \mathrm{L} = -\frac{r}{s}$, and the other coefficients vanish. Hence b, b', c, c' all vanish, by (10), and the condition (11) is satisfied.

The equations (13) become

$$\frac{du}{dp} + \frac{t}{s} \cdot \frac{du}{dq} = 0,$$

$$\frac{r}{s} \cdot \frac{du}{dp} + \frac{du}{dq} = 0,$$

which require $\frac{du}{dp} = 0, \quad \frac{du}{dq} = 0.$

The equations (12), (14) become on reduction

$$\frac{du}{dx} = 0, \quad \frac{du}{dy} = 0, \quad \frac{du}{dz} = 0,$$

$$s\frac{du}{ds} + r\frac{du}{dr} + t\frac{du}{dt} = 0.$$

The common solutions of this system are given by the ordinary differential equations

$$sdr - rds = 0,$$

$$sdt - tds = 0,$$

whence $$u_1 = \frac{r}{s}, \quad u_2 = \frac{s}{t}.$$

The required first integral is therefore

$$\mathrm{F}\left\{\frac{r}{s}, \frac{s}{t}\right\} = 0.$$

Class I. (2) *When* K *vanishes.*

16. If K vanish, the values of b, b', c, c' given in (10) are much simplified. The condition (11) has no longer any significance. In fact, it is derived from (ii.), (iv.), (v.) of (8), which reduce to two equations only when K vanishes. To find what replaces (11) in the present case, it is only necessary to replace b, b', c', in (i.) of (8), by their values when K vanishes. The result is

$$\mathrm{A^2L^3 + ACL^2H + BDH^2L + D^2H^3 = H^2L^2M} \quad \ldots\ldots\ldots\ldots (15).$$

Since $\mathrm{K^2 - HL}$ does not vanish, and K is zero, neither H nor L vanish. The first integrals of an equation of this group may be found either from (iv.), (vii.), and (x.), or from (vi.), (ix.), and (x.) of (8); and it is convenient to assume H or L respectively to be unity, as may be done without loss of generality.

From either set of equations four more can be derived corresponding to (13), (14). If these be mutually independent, there is no first integral of the assumed form (1); but if they are equivalent to only three equations, then there may be a first integral. The process, however, will not in any case enable us to carry the integration further.

17. *Example* 3.

$$t(a\gamma - \beta^2) - r(\beta\delta - \gamma^2) + sta - t(r+1)\beta + s(r+1)\gamma - r^2\delta + s^2 - rt = 0 \ldots (\mathrm{i.})$$

On trial it will be found that the condition (15) is satisfied.

Divide the equation throughout by t, so that H may be unity. Then, from (10), putting $\mathrm{K} = 0$, we get

$$b = r, \quad c' = \frac{s}{t},$$

$$b' = 1, \quad c = \frac{sr}{t}.$$

The equations (iv.), (vii.), (x.) of (8) become in this case

$$\frac{du}{dx} + p\frac{du}{dz} + r\frac{du}{dp} + s\frac{du}{dq} - \frac{s}{t}\cdot\frac{du}{dr} - r\frac{du}{ds} = 0 \quad \ldots\ldots\ldots \text{(ii.)},$$

$$\frac{du}{dy} + q\frac{du}{dz} + s\frac{du}{dp} + t\frac{du}{dq} - \frac{du}{dr} - s\frac{du}{ds} = 0 \quad \ldots\ldots\ldots \text{(iii.)},$$

$$\frac{du}{dt} - \frac{r}{t}\cdot\frac{du}{dr} = 0 \quad \ldots\ldots\ldots \text{(iv.)}$$

The equations derived from these by Boole's rule are

$$\frac{du}{dq} - \frac{1}{t}\cdot\frac{du}{dr} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(v.)},$$

$$\frac{du}{dp} - \frac{du}{ds} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(vi.)},$$

$$(r-1)\frac{du}{dp} - s\frac{du}{dq} + \frac{s}{t}\cdot\frac{du}{dr} + (1-r)\frac{du}{ds} = 0,$$

of which the third is a consequence of the other two.

From (vi.) and (ii.) another equation can be formed, which reduces, by means of (v.), to

$$\frac{du}{dz} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(vii.)}$$

The six equations (ii.) to (vii.) may be put into the form

$$\frac{du}{dx} = 0, \quad \frac{du}{dy} = 0, \quad \frac{du}{dz} = 0,$$

$$t\frac{du}{dt} - r\frac{du}{dr} = 0,$$

$$t\frac{du}{dq} - \frac{du}{dr} = 0,$$

$$\frac{du}{dp} - \frac{du}{ds} = 0;$$

and the common solutions are given by integrating

$$tdr + rdt + dq = 0,$$

$$ds + dp = 0;$$

viz., these solutions are

$$u_1 = rt + q,$$

$$u_2 = s + p.$$

The first integral of (i.) is therefore

$$\mathrm{F}(rt + q, s + p) = 0.$$

CLASS II. $K^2-HL=0$; *but* H, K, L *do not all vanish.*

18. It is an immediate consequence of the relations by which this case is defined that H, L cannot both vanish. Furthermore, it is only necessary to discuss the equation in which H does not vanish; for the excluded case in which $H=0$ (and therefore $K=0$, $L\neq 0$) is just the same as that in which $H\neq 0$, $K=0$, $L=0$, save that the two independent variables x, y have been interchanged.

On the assumption that H does not vanish, it may be replaced by unity, and then we have

$$L=K^2.$$

Thus the three coefficients H, K, L may be replaced by 1, K, K^2, and under this form the defining equations of Class II. are implied, so that the explicit statement of them is no longer required.

19. *Determination of b, &c.*—Referring, for a moment, to paragraph 10, it will be seen that the values of b, &c. become infinite when $K^2=HL$, unless

$$AL^3+BKL^2+CK^2L+DK^3=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (15);$$

and that when this condition is fulfilled they become indeterminate. The reason for this is tolerably obvious. The values of b, c', say, are deduced from two equations (9). The condition that (when $K^2=HL$) these equations should be consistent is (15); but when they are consistent they become identical, and are therefore insufficient for the determination of the two unknowns b, c'. It is necessary then, in this case, to take at least one more of the equations (6), (8) into account. We re-write these equations here, putting, as suggested above, 1 for H, K^2 for L.

$$\left.\begin{aligned} b+b'&=B\,;\quad c+c'=C \quad \ldots\ldots\ldots (i.), (ii.)\\ bb'&=Ac'-M \quad \ldots\ldots\ldots\ldots\ldots (iii.)\\ bc&=AD+KM \quad \ldots\ldots\ldots\ldots\ldots (iv.)\\ cc'&=Db'-K^2M \quad \ldots\ldots\ldots\ldots (v.)\\ K^2b+Kc'+D&=0 \quad \ldots\ldots\ldots\ldots (vi.)\\ Kb'+c+K^2A&=0 \quad \ldots\ldots\ldots\ldots (vii.) \end{aligned}\right\} \ldots\ldots (16).$$

These are not all independent, the last five being equivalent to three only. The whole system is therefore sufficient to determine the four quantities b, b', c, c', and one relation between K, A, B, C, D, M. The latter is obtained by adding (vii.) multiplied by K to (vi.), and using (i.), (ii.) The result is

$$AK^3+BK^2+CK+D=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (17),$$

which is the same as (15).

An equation for b is found by eliminating b', c' from (i.), (iii.), (vi.) of (16), the result being

$$b^3-(\mathrm{B}+\mathrm{AK})\,b-\frac{\mathrm{AD}+\mathrm{MK}}{\mathrm{K}}=0 \quad \ldots\ldots\ldots\ldots\ldots (18).$$

Hence b may have either of two values; but when the value of b is fixed, the quantities b', c, c' are determined without ambiguity by the equations (i.), (iii.), (ii.) respectively. Thus, then, we have two sets of values, which will be indicated, when necessary, by b_1, b_1', c_1, c_1'; and b_2, &c. To each of these systems will correspond a system of equations for a first integral; and therefore an equation (2) of the kind we are now considering may have two first integrals.

20. The three equations of (7) which serve to find the first integrals are (iv.), (vii.), (x.), which become, when $\mathrm{H}=1$, $\mathrm{L}=\mathrm{K}^2$,

$$\left.\begin{aligned}
\Delta_1 u \equiv \frac{du}{dx}+p\frac{du}{dz}+r\frac{du}{dp}+s\frac{du}{dq}-c'\frac{du}{dr}-b\frac{du}{ds}&=0\\
\Delta_2 u \equiv \frac{du}{dy}+q\frac{du}{dz}+s\frac{du}{dp}+t\frac{du}{dq}-b'\frac{du}{dr}-\mathrm{A}\frac{du}{ds}&=0\\
\Delta_3 u \equiv \frac{du}{dt}\qquad\qquad\qquad\qquad+\mathrm{K}^2\frac{du}{dr}-\mathrm{K}\frac{du}{ds}&=0
\end{aligned}\right\} \ldots\ldots (19).$$

The common solutions of these equations satisfy the equations

$$\left.\begin{aligned}
\frac{du}{dq}-\quad \mathrm{K}\frac{du}{dp}-(\Delta_3 b'+\Delta_2\mathrm{K}^2)\frac{du}{dr}-(\Delta_3\mathrm{A}-\Delta_2\mathrm{K})\frac{du}{ds}&=0\\
\mathrm{K}\frac{du}{dq}-\quad \mathrm{K}^2\frac{du}{dp}+(\Delta_1\mathrm{K}^2+\Delta_3 c')\frac{du}{dr}-(\Delta_1\mathrm{K}-\Delta_3 b)\frac{du}{ds}&=0\\
\mathrm{A}\frac{du}{dq}-(b-b')\frac{du}{dp}+(\Delta_2 c'-\Delta_1 b')\frac{du}{dr}+(\Delta_2 b-\Delta_1\mathrm{A})\frac{du}{ds}&=0
\end{aligned}\right\} \ldots\ldots (20).$$

$$\frac{du}{dz}-\quad \mathrm{P}\frac{du}{dp}-\ \&\mathrm{c}.\qquad\qquad =0 \ldots\ldots (21).$$

Of the seven equations thus formed, (19), (21) are mutually independent, for the latter does not contain $\frac{du}{dx}$, $\frac{du}{dy}$, or $\frac{du}{dt}$. Also (20) are not algebraical results of (19), (21), inasmuch as they do not contain $\frac{du}{dx}$, $\frac{du}{dy}$, $\frac{du}{dz}$, or $\frac{du}{dt}$. If then the three equations (20) be also mutually independent, there will be a system of seven independent equations, and these cannot have more than one common solution. Accordingly there can be no first integral of the form (1). For the discussion of the cases in which (20) are not mutually independent, it is convenient to take, in turn, the cases in which the third equation of (20) is or is not a consequence of the other two.

21. One of the conditions that the third equation of (20) should be a consequence of the other two—viz., the condition derived from considering $\frac{du}{dp}, \frac{du}{dq}$ only—is

$$b-b' = \mathrm{AK} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22).$$

But $$b+b' = \mathrm{B},$$

so that $$2b = \mathrm{B}+\mathrm{AK}, \quad 2b' = \mathrm{B}-\mathrm{AK}.$$

Now from (iii.), (vi.) of (16) we get, by eliminating c',

$$\mathrm{K}bb'+\mathrm{KM}+\mathrm{AK}^2b+\mathrm{AD} = 0,$$

whence, by (22), $$\mathrm{K}b^2+\mathrm{KM}+\mathrm{AD} = 0.$$

If in this for b we put its value just found, we find

$$\mathrm{K}(\mathrm{B}+\mathrm{AK})^2+4(\mathrm{KM}+\mathrm{AD}) = 0.$$

But this is the condition that (18) should have equal roots. In this case, then, the two first integrals are identical.

22. If the first two equations of (20) should be mutually independent, and the supposition of the last paragraph still hold good, the system (19), (20), (21) comprises at least six independent equations; and there cannot be more than two common solutions of the system. In this case we may be able to find one first integral, or rather two identical first integrals; but the process cannot be carried farther. On the other hand, if the two first equations of (20) reduce to one only, in which is included the third of (20), then the system (19) may have three common solutions. Of these, two will give the first integral, and the third will give a second integral of the proposed equation.

23. *Example* 4.

$$\alpha\gamma-\beta^2+\mathrm{K}(\alpha\delta-\beta\gamma)+\mathrm{K}^2(\beta\delta-\gamma^2)+\frac{s+\mathrm{K}t}{y}\alpha+\frac{\mathrm{K}s+\mathrm{K}^2t}{y}\beta$$
$$+\frac{r+\mathrm{K}s}{x}\gamma+\frac{\mathrm{K}r+\mathrm{K}^2s}{x}\delta+\frac{rs+\mathrm{K}(rt+s^2)+\mathrm{K}^2st}{xy} = 0$$

(K constant).

Here B+AK, AD+KM both vanish; also CK+D vanishes, so that (17) is satisfied. From (18) we get

$$b_1 = b_2 = 0;$$

and then, from (16),

$$b' = -\frac{\mathrm{K}s+\mathrm{K}^2t}{y},$$
$$c = 0,$$
$$c' = \frac{r+\mathrm{K}s}{x}.$$

In this case the equations (19), (20), (21) are easily reduced to

$$\frac{du}{dx}+(r+\mathrm{K}s)\frac{du}{dp}-\frac{r+\mathrm{K}s}{x}\cdot\frac{du}{dr}=0,$$

$$\frac{du}{dy}+(s+\mathrm{K}t)\frac{du}{dp}-\frac{s+\mathrm{K}^2t}{y}\cdot\frac{du}{dr}-\frac{s+\mathrm{K}t}{y}\cdot\frac{du}{ds}=0,$$

$$\frac{du}{dt}+\mathrm{K}^2\frac{du}{dr}-\mathrm{K}\frac{du}{ds}=0,$$

$$\frac{du}{dq}-\mathrm{K}\frac{du}{dp}=0,$$

$$\frac{du}{dz}=0,$$

The common solutions of this system are given by

$$dp-(r+\mathrm{K}s)\,dx-(s+\mathrm{K}t)\,dy+\mathrm{K}dq=0,$$

$$dr+(r+\mathrm{K}s)\frac{dx}{x}+(\mathrm{K}s+\mathrm{K}^2t)\frac{dy}{y}-\mathrm{K}^2dt=0,$$

$$ds+(s+\mathrm{K}t)\frac{dy}{y}+\mathrm{K}dt=0,$$

of which the second may be written, on account of the third,

$$dr+(r+\mathrm{K}s)\frac{dx}{x}+\mathrm{K}ds=0,$$

The integrals of these equations are

$$(r+\mathrm{K}s)\,x=c_1,$$

$$(s+\mathrm{K}t)\,y=c_2,$$

$$p+\mathrm{K}q-c_1\log x-c_2\log y=c_3.$$

Hence the first integral is

$$\mathrm{F}\{x(r+\mathrm{K}s),\,y(s+\mathrm{K}t)\}=0,$$

and the second integral is given by the elimination of c between the equations

$$p+\mathrm{K}q=c+\log x\,.\,\phi(c)+\log y\,.\,\psi(c),$$

$$0=1+\log x\,.\,\phi'(c)+\log y\,.\,\psi'(c).$$

24. *Example* 5.

$$q^2(a\gamma-\beta^2)+pq(a\delta-\beta\gamma)+p^2(\beta\delta-\gamma^2)$$
$$+2qsta+2(pst-qs^2-qrt)\beta+2(qrs-ps^2-prt)\gamma+2prs\delta$$
$$-(s^2-rt)^2=0\ \ldots(\mathrm{i.})$$

Here $\mathrm{K}=\frac{p}{q}$, $\mathrm{B}+\mathrm{AK}=2\frac{s^2+rt}{q}\cdot\frac{\mathrm{AD}}{\mathrm{K}}+\mathrm{M}=-\frac{(s^2+rt)^2}{q^2}$,

so that

$$b = \frac{s^2+rt}{q},$$

$$b' = -\frac{2pst}{q^2} + \frac{s^2+rt}{q},$$

$$c = \frac{p}{q^2}(s^2+rt),$$

$$c' = \frac{2rs}{q} - \frac{p(s^2+rt)}{q^2}.$$

In this case, then, the equations (20) become

$$\frac{du}{dq} - \frac{p}{q}\cdot\frac{du}{dp} - \frac{q^2r-2pqs+p^2t}{q^3}\cdot\frac{du}{dr} - \frac{qs-pt}{q^2}\cdot\frac{du}{ds} = 0.$$

Using this, (19) become

$$\frac{du}{dx} + p\frac{du}{dz} + \frac{qr+ps}{q}\cdot\frac{du}{dp} - \frac{(qr+ps)(qs-pt)}{q^3}\cdot\frac{du}{dr} - \frac{(qr+ps)t}{q^2}\cdot\frac{du}{ds} = 0,$$

$$\frac{du}{dy} + q\frac{du}{dz} + \frac{qs+pt}{q}\cdot\frac{du}{dp} - \frac{(qs+pt)(qs-pt)}{q^3}\cdot\frac{du}{dr} - \frac{(qs+pt)t}{q^2}\cdot\frac{du}{ds} = 0,$$

$$\frac{du}{dt} + \frac{p^2}{q^2}\cdot\frac{du}{dr} - \frac{p}{q}\cdot\frac{du}{ds} = 0.$$

As the derivation of another equation from these four is somewhat tedious, we may at once take the ordinary differential equations corresponding to these four equations, and modify them as necessary. The ordinary equations are

$$dz - p\,dx - q\,dy = 0,$$

$$q\,dp - (qr+ps)\,dx - (qs+pt)\,dx + p\,dq = 0 \quad \ldots\ldots\ldots \text{(ii.)},$$

$$q^3dr + (q^2r-2pqs+p^2t)\,dq + (qr+ps)(qs-pt)\,dx + (qs+pt)(qs-pt)\,dy - p^2q\,dt = 0 \quad \ldots\text{(iii.)},$$

$$q^2ds + (qs-pt)\,dq + (qr+ps)\,t\,dx + (qs+pt)\,t\,dy + rq\,dt = 0 \quad \ldots \text{(iv.)}$$

Adding (ii.) multiplied by t to (iv.), and dividing throughout by q, we get

$$q\,ds + s\,dq + t\,dp + p\,dt = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(v.)},$$

whence

$$u_1 = qs+pt.$$

Again, forming the equation

$$(qs-pt)\text{ (ii.)} + \text{(iii.)} + pq\text{ (v.)} = 0,$$

and taking out the common factor q^3, we get

$$q\,dr + r\,dq + p\,ds + s\,dp = 0,$$

whence

$$u_2 = qr+ps.$$

Finally, substituting these results in (ii.),

$$q\,dp + p\,dq = u_2\,dx + u_1\,dy.$$

Hence the first integral of the given equation is

$$\mathrm{F}(qr+ps,\ qs+pt) = 0,$$

and the second integral is the result of eliminating c between the equations

$$pq = c+x\,\phi(c)+y\,\psi(c),$$
$$0 = 1+x\,\phi'(c)+y\,\psi'(c).$$

25. If the two first integrals be not identical, the third equation of (20) is necessarily independent of the first two. In order that there may be a first integral, these two equations must reduce to one only. We then have two sets of equations, each of which can at most have two common solutions. Hence each of these systems taken separately may give a first integral; but the integration cannot be carried further. But there is another possibility here, suggested by the analogy of the equations of the second order. It is possible that by one of these first integrals we may be able to integrate the other. This is certainly the case when the equation of the third is virtually an equation of the second order only, K being zero; and the following example will show that the process will succeed in some cases in which K does not vanish.

26. *Example* 6.

$$\alpha\gamma-\beta^2+\mathrm{K}(\alpha\delta-\beta\gamma)+\mathrm{K}^2(\beta\delta-\gamma^2)+\mathrm{A}\alpha-(\mathrm{B}'-\mathrm{AK})\beta-\mathrm{B}'\mathrm{K}\gamma = 0,$$

where K, A, B′ are constants.

The equation for b in this case is

$$b^2-\mathrm{B}'b = 0;$$
$$\therefore\quad b_1 = 0,\quad b_2 = \mathrm{B}'.$$

Also

$$b_1' = \mathrm{B}'-\mathrm{AK},\quad b_2' = -\mathrm{AK},$$
$$c_1' = 0,\qquad c_2' = -\mathrm{B}'\mathrm{K}.$$

The equations (20), (21) reduce to

$$\frac{du}{dp} = 0,\quad \frac{du}{dq} = 0,\quad \frac{du}{dz} = 0$$

in each case.

The equations (19) become, for the first integral corresponding to $b_1, \ldots,$

$$\frac{du}{dx} = 0,$$
$$\frac{du}{dy}-(\mathrm{B}'-\mathrm{AK})\frac{du}{dr}-\mathrm{A}\frac{du}{ds} = 0,$$
$$\frac{du}{dt}+\mathrm{K}^2\frac{du}{dr}-\mathrm{K}\frac{du}{ds} = 0.$$

The ordinary equations are easily reduced to

$$dr+\mathrm{K}ds+\mathrm{B}'dy=0,$$
$$ds+\mathrm{K}\,dt+\mathrm{A}\,dy=0,$$

giving

$$u_1=r+\mathrm{K}s+\mathrm{B}'y,$$
$$u_2=s+\mathrm{K}t+\mathrm{A}\,y,$$

so that one first integral is

$$\mathrm{F}\{r+\mathrm{K}s+\mathrm{B}'y,\ s+\mathrm{K}t+\mathrm{A}y\}=0 \quad \text{.....(i.)}$$

For the other first integral we have

$$\frac{du}{dx}+\mathrm{B}'\mathrm{K}\frac{du}{dr}-\mathrm{B}'\frac{du}{ds}=0,$$
$$\frac{du}{dy}+\mathrm{A}\mathrm{K}\frac{du}{dr}-\mathrm{A}\frac{du}{ds}=0,$$
$$\frac{du}{dt}+\mathrm{K}^2\frac{du}{dr}-\mathrm{K}\frac{du}{ds}=0,$$

which give without difficulty

$$v_1=r+\mathrm{K}s,$$
$$v_2=s+\mathrm{K}t+\mathrm{B}'x+\mathrm{A}y,$$

so that another first integral is

$$\mathrm{F}\{r+\mathrm{K}s,\ s+\mathrm{K}t+\mathrm{B}'x+\mathrm{A}y\}=0 \quad \text{.....(ii.)}$$

Now, although taken separately neither of these equations is further integrable, yet taken together they make the equation

$$dp+\mathrm{K}\,dq=(r+\mathrm{K}s)\,dx+(s+\mathrm{K}t)\,dy \quad \text{..... (iii.)},$$

an exact differential, and thus permit another integration.

For instance, $s+\mathrm{K}t=u_2-\mathrm{A}y.$

Hence, by (ii.), $r+\mathrm{K}s=\chi\,(u_2+\mathrm{B}'x).$

Substituting these values in (iii.), integrating, and putting a function of u_2 for the arbitrary constant, we get a "complete" second integral, viz.,

$$p+\mathrm{K}q=\phi\,(u_2+\mathrm{B}'x)+u_2\,y-\frac{\mathrm{A}}{2}\,y^2+\psi\,(u_2).$$

The general second integral is found by elimination of u_2 between this equation and

$$0=\phi'(u_2+\mathrm{B}'x)+y+\psi'(u_2);$$

or we may specify the forms of F, in (i.), (ii.); thence get values of $r+\mathrm{K}s$, $s+\mathrm{K}t$, the substitution of which will make (iii.) an exact differential.

27. Before passing to the linear equation, an interesting point claims notice. From the third equation of (19) and the first of (20) we can derive another equation which is satisfied by all the common solutions of (19). This is of the form

$$\Delta_3 \mathrm{K} \cdot \frac{du}{dp} + \mathrm{R}\frac{du}{dr} + \mathrm{S}\frac{du}{ds} = 0,$$

R, S being written for the complicated coefficients of $\frac{du}{dr}$, $\frac{du}{ds}$. Now, if the equation of the third order be integrable, this new equation must be an identity, or a consequence of (20). In the former case we require

$$\Delta_3 \mathrm{K}, \equiv \frac{d\mathrm{K}}{dt} - \mathrm{K}\frac{d\mathrm{K}}{ds} + \mathrm{K}^2\frac{d\mathrm{K}}{dr}, = 0 \ldots\ldots\ldots\ldots\ldots (23).$$

I am inclined to think that this relation is a necessary condition of integrability, but do not see my way to a simple proof. At any rate, (23) defines an important group of equations—including, for instance, all cases in which K does not involve r, s, t—which are distinguished by some notable properties. One of these properties we shall here indicate.

Integrating (23), we find

$$\mathrm{K} = \phi\,(r + \mathrm{K}s,\ s + \mathrm{K}t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (24);$$

and then the equation

$$\Delta_3 u, \equiv \frac{du}{dt} - \mathrm{K}\frac{du}{ds} + \mathrm{K}^2\frac{du}{dr}, = 0 \quad \ldots\ldots\ldots (19, 3)$$

(which is satisfied by both first integrals) gives

$$u = \phi\,(\mathrm{K},\ r + \mathrm{K}s,\ s + \mathrm{K}t) \ldots\ldots\ldots\ldots\ldots\ldots (25),$$

as far as r, s, t are concerned. Of the three arguments in (25) only two are independent on account of (24).

Let us suppose, for simplicity, that $r + \mathrm{K}s$, $s + \mathrm{K}t$ are independent.* If we can find two solutions of (19), we shall have $r + \mathrm{K}s$, $s + \mathrm{K}t$ expressed in terms of p, q, &c., and two constants of integration. The same would be true of K in virtue of (24). Thus, in the equation

$$dp + \mathrm{K}dq = (r + \mathrm{K}s)\,dx + (s + \mathrm{K}t)\,dy \ldots\ldots\ldots\ldots (25'),$$

the coefficients K, &c. do not involve r, s, t. When the two first integrals are identical, it appears that (25′) becomes an exact differential. When, on the other hand, there are two different first integrals, we have two equations like (25) from which we may be able to form an integrable combination.

These remarks are illustrated by the following examples :—

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* This will be the case, for example, if $\mathrm{K} = \frac{r}{q-s}$, but not if $\mathrm{K} = \frac{s-r}{s-t}$.

28. *Example* 7.

$$(s-t)^2(a\gamma-\beta^2)+(s-r)(s-t)(a\delta-\beta\gamma)+(s-r)^2(\beta\gamma-\delta^2)=0.$$

Here $$b=b'=c=c'=0,$$

$$\mathrm{K}=\frac{s-r}{s-t},$$

which satisfies the relation (23). The equations (19), (20) become

$$\left(\frac{du}{dx}\right)=0, \quad \left(\frac{du}{dy}\right)=0,$$

$$\frac{du}{dt}+\left(\frac{s-r}{s-t}\right)^2\cdot\frac{du}{dr}-\frac{s-r}{s-t}\cdot\frac{du}{ds}=0,$$

$$\frac{du}{dq}-\frac{s-r}{s-t}\cdot\frac{du}{dp}=0.$$

Instead of solving these equations, as before, we may notice that

$$r+\mathrm{K}s=\frac{s^2-rt}{s-t}=s+\mathrm{K}t.$$

Hence, for the common solutions of the above system, we may put

$$(\mathrm{K}=)\ \frac{s-r}{s-t}=u_1,$$

$$\frac{s^2-rt}{s-t}=u_2,$$

which can easily be verified.

The first integrals are identical, viz.,

$$\mathrm{F}\left\{\frac{s-r}{s-t},\ \frac{s^2-rt}{s-t}\right\}=0.$$

For the second integral we must integrate

$$dp+u_1dq=u_2(dx+dy),$$

whence $$x+y=u_3+\frac{1}{u_2}\cdot p+\frac{u_1}{u_2}q.$$

The general second integral is therefore given by eliminating u between

$$x+y=u+\phi(u)\cdot p+\psi(u)\cdot q,$$

$$0=1+\phi'(u)\cdot p+\psi'(u)\cdot q.$$

29. *Example* 8.

$$a\gamma-\beta^2+\frac{r}{a-s}(a\delta-\beta\gamma)+\left(\frac{r}{a-s}\right)^2(\beta\delta-\gamma^2)$$
$$+\frac{\lambda r}{a-s}a+\left\{\frac{\lambda r^2}{(a-s)^2}-\frac{\mu r}{a-s}\right\}\beta-\frac{\mu r^2}{(a-s)^2}\gamma=0 \quad \ldots\ldots \text{(i.)}$$

Here $\mathrm{K}=\frac{r}{a-s}$; $\mathrm{A}=\lambda\mathrm{K}$; $\mathrm{B}=\mu\mathrm{K}-\lambda\mathrm{K}^2$; $\mathrm{C}=-\mu\mathrm{K}^2$,

and (17) is satisfied. The equation (18) for b becomes

$$b^2-\mu\mathrm{K}b=0,$$

whence, with the help of (16), we get

$$b_1 = \mu K, \quad b_1' = -\lambda K^2, \qquad c_1 = 0, \qquad c_1 = -\mu K^2,$$
$$b_2 = 0, \quad b_2' = \mu K - \lambda K^2, \quad c_2 = -\mu K^2, \quad c_2' = 0.$$

Taking the first set of values,

$$\left.\begin{aligned} \Delta_1 u &\equiv \left(\frac{du}{dx}\right) + \mu K^2 \frac{du}{dr} - \mu K \frac{du}{ds} = 0 \\ \Delta_2 u &\equiv \left(\frac{du}{dy}\right) + \lambda K^2 \frac{du}{dr} - \lambda K \frac{du}{ds} = 0 \\ \Delta_3 u &\equiv \frac{du}{dt} + K^2 \frac{du}{dr} - K \frac{du}{ds} = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots \text{(ii.)}$$

To form (20), we notice that

$$\Delta_1 K = 0, \quad \Delta_2 K = 0, \quad \Delta_3 K = 0;$$

whence it follows that the expressions $\Delta_1 b_2$, &c. all vanish. Thus (20) become

$$\frac{du}{dq} + K\frac{du}{dp} = 0, \quad \lambda K \frac{du}{dq} - (\mu K + \lambda K^2)\frac{du}{dp} = 0,$$

or

$$\frac{du}{dp} = 0, \quad \frac{du}{dq} = 0.$$

To these we add (21), which is simply

$$\frac{du}{dz} = 0,$$

and then in (ii.) we may omit the brackets from $\left(\frac{du}{dx}\right)$, $\left(\frac{du}{dy}\right)$.

The solutions we get are

$$\left.\begin{aligned} \frac{r}{a-s}, &= K, \quad = u_1 \\ \frac{as-(s^2-rt)}{r} + \mu x + \lambda y, &= \frac{s+Kt}{K} + \mu x + \lambda y, = u_2 \end{aligned}\right\} \ldots\ldots \text{(iii.)}$$

Corresponding to the second set of values, we can find no solutions. From (iii.) we get

$$K = u_1; \; r + Ks = r + \frac{rs}{a-s} = \frac{ar}{a-s} = aK = au_1,$$

$$s + Kt = K(u_2 - \mu x - \lambda y) = u_1(u_2 - \mu x - \lambda y).$$

Hence (25′) becomes

$$dp + u_1 dq - au_1 dx - u_1(u_2 - \mu x - \lambda y)\, dy = 0 \ldots\ldots\ldots \text{(iv.)}$$

Now this is not an exact differential, and cannot be made so by any factor. As we cannot find the other first integral, the integration cannot be continued any further. But notice that, if μ vanishes, (iv.) is an exact differential; so that we appear to be able to get a second integral in this case, although we only know one first integral. But if μ vanishes, the two sets of values for b, b', &c. become identical, we have, in fact, two identical first integrals, and it is interesting to find that the change of (iv.) to an exact differential occurs just as the two first integrals of (i.) become identical.

CLASS III. *Linear Equations.*

30. We have now to discuss the linear equation

$$A\alpha - B\beta + C\gamma - D\delta + M = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (26).$$

The equations (8) for the determination of b, &c., become

$$\left.\begin{aligned} bb' &= Ac' \quad \ldots\ldots (i.) \\ bc &= AD \quad \ldots\ldots (ii.) \\ cc' &= Db' \quad \ldots\ldots (iii.) \end{aligned}\right\} \ldots\ldots\ldots\ldots (27).$$

and, as before,

$$\left.\begin{aligned} b + b' &= B \quad \ldots\ldots (iv.) \\ c + c' &= C \quad \ldots\ldots (v.) \end{aligned}\right\}$$

Of these (iii.) is clearly a consequence of (i.), (ii.) Hence there are four equations to determine four unknowns, or we shall get no equation of condition between A, B, &c. There is no difficulty in finding the following equation for b:

$$b^3 - Bb^2 + AC \cdot b - A^2D = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (28).$$

When the value of b is determined, b', c, c' are determined without ambiguity, by (iv.), (i.), (v.) Hence there are three auxiliary systems of equations each of which may determine a first integral. Of course any two or all of these systems may become identical, and any one of them may not admit of a common solution.

31. It is convenient to take two cases of the linear equation for discussion. In the first place we assume that A does not vanish. The results then obtained are of course applicable to the case in which D does not vanish: the application only supposes an interchange of the two independent variables.

The cases excluded from the first group are those in which A, D both vanish. We need only discuss these equations on the assumption that B does not vanish, for the results thus obtained may be applied directly to the only neglected case, viz., that in which B, but not C, vanishes.

(1). *A, D do not both vanish.*

32. We assume that A does not vanish, and put it equal to unity. The equation (28) becomes

$$b^3 - Bb^2 + Cb - D = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (29),$$

while b', c, c' are given by

$$\left.\begin{aligned} b' &= B - b \\ c &= \frac{D}{b} \\ c' &= C - \frac{D}{b} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (30).$$

As before, we shall indicate the values of b', &c., for different roots of (29), by corresponding subscripts, 1, 2, 3.

The equations for giving the first integral are (i.), (vii.), (viii.) of (7), viz.,

$$\left.\begin{aligned} \Delta_1 u \equiv \frac{du}{dx} - b\frac{du}{dy} + (p - bq)\frac{du}{dz} + (r - bs)\frac{du}{dp} + (s - bt)\frac{du}{dq} - \mathrm{M}\frac{du}{dr} &= 0 \\ \Delta_2 u \equiv \frac{du}{ds} \qquad\qquad\qquad\qquad + b'\frac{du}{dr} &= 0 \\ \Delta_3 u \equiv \frac{du}{dt} \qquad\qquad\qquad\qquad - c\ \frac{du}{dr} &= 0 \end{aligned}\right\} \quad \ldots\ (31).$$

From these we may form the derived system

$$\left.\begin{aligned} (\Delta_3 b' + \Delta_2 c)\frac{du}{dr} &= 0 \\ \Delta_3 b \,.\, \left(\frac{du}{dy}\right) + c\frac{du}{dp} + b\frac{du}{dq} - (\Delta_3 \mathrm{M} + \Delta_1 c)\frac{du}{dr} &= 0 \\ \Delta_2 b \,.\, \left(\frac{du}{dy}\right) + (b - b')\frac{du}{dp} - du - (\Delta_2 \mathrm{M} - \Delta_1 b')\frac{du}{dr} &= 0 \end{aligned}\right\} \quad \ldots\ldots (32).$$

Now $\frac{du}{dr}$ cannot identically vanish; for then, by the last two equations of (31), $\frac{du}{ds}$, $\frac{du}{dt}$ would also vanish identically, and there would be no proper first integral. Hence, from the first of (32), we have

$$\Delta_3 b' + \Delta_2 c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (33).$$

Replacing b', c by their values in (30), and eliminating b, $\Delta_2 b$, $\Delta_3 b$ from the equation thus formed by means of (29) and the two equations derived from (29) by acting on it with Δ_2, Δ_3, we get a relation which is necessarily satisfied by B, C, D if there be a first integral.

33. Suppose the three roots of (29) to be equal. It will then be found that the five equations (31), (32) reduce to four. For we have

$$\mathrm{B} = 3b, \quad \mathrm{C} = 3b^2, \quad \mathrm{D} = b^3,$$
$$b' = 2b, \quad c = b^2, \quad c' = 2b^2.$$

The equations (31) become

$$\left.\begin{aligned} \left(\frac{du}{dx}\right) - b\left(\frac{du}{dy}\right) - \mathrm{M}\,.\,\frac{du}{dr} &= 0 \\ \frac{du}{ds} \qquad\qquad + 2b\,.\,\frac{du}{dr} &= 0 \\ \frac{du}{dt} \qquad\qquad - b^2\,.\,\frac{du}{dr} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ (34).$$

The relation (33) becomes

$$\frac{db}{d} + b\frac{db}{ds} + b^2\frac{db}{dr} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ (35),$$

while (32) are

$$\left.\begin{aligned}\Delta_3 b\,.\left(\frac{du}{dy}\right)+b^2\frac{du}{dp}+b\frac{du}{dq}-(\Delta_3 \mathrm{M}+2b\Delta_1 b)\frac{du}{dr}=0\\ \Delta_2 b\,.\left(\frac{du}{dy}\right)-b\frac{du}{dp}-\frac{du}{dq}-(\Delta_2 \mathrm{M}-2\Delta_1 b)\frac{du}{dr}=0\end{aligned}\right\}\ldots\ldots\ldots(36).$$

Adding the second of these multiplied by b to the first, we get, in virtue of (35),

$$(\Delta_3 \mathrm{M}+b\Delta_2 \mathrm{M})\,.\frac{du}{dr}=0,$$

of which only the first factor can vanish. Hence

$$\frac{d\mathrm{M}}{dt}+b\frac{d\mathrm{M}}{ds}+b^2\frac{d\mathrm{M}}{dt}=0\ldots\ldots\ldots\ldots\ldots\ldots(37),$$

and the two equations (36) become identical. If no more equations could be formed from (34), (36), there would be four common solutions of (34), which would give two first integrals, a second integral, and a third integral. If, however, as seems to be generally the case, another equation can be formed, the third integral is not obtainable.

34. *Example* 9.

$$\alpha-3b\beta+3b^2\gamma-b^3\gamma=0,$$

where b is constant.

The equations (36) become

$$\frac{du}{dq}+b\frac{du}{dp}=0,$$

and by this the other equations are reduced to

$$\frac{du}{dx}-b\frac{du}{dy}+(p-bq)\frac{du}{dz}+(r-2bs+b^2t)\frac{du}{dp}=0,$$

$$\frac{du}{ds}+2b\frac{du}{dr}=0,$$

$$\frac{du}{dt}-b^2\frac{du}{dr}=0.$$

The ordinary equations formed from these are

$$\begin{aligned}dr-2bds+b^2dt&=0,\\ dy+bdx&=0,\\ dp-bdq-(r-2bs+b^2t)\,dx&=0,\\ dz-(p-bq)\,dx&=0.\end{aligned}$$

From these we get

$$\begin{aligned}r-2bs+b^2t&=u_1,\\ y+bx&=u_2,\end{aligned}$$

giving for first integral

$$\mathrm{F}\{r-2bs+b^2t,\ y+bx\}=0.$$

The third equation gives $p - bq = u_1 x + u_3$,

and then the fourth $z = \frac{u_1}{2} x^2 + u_3 x + u_4$.

Here u_3, u_4 are functions of u_1, which is itself a function of u_2 or $y + bx$. The general second and third integrals may therefore be written

$$p - bq = \varphi(y + bx) + x\psi(y + bx),$$
$$z = \phi(y + bx) + x\psi(y + bx) + x^2\chi(y + bx).$$

35. When the roots of (29) are not all equal, the integration cannot be completed from any one of the systems (31) taken separately. But it may be possible to find, from any first integrals that have been determined, values of r, s, t which will make

$$dp = r\,dx + s\,dy,$$
$$dq = s\,dx + t\,dy$$

exact differentials; and finally, from the result of this, to find p, q, so that

$$dz = p\,dx - q\,dy$$

may be an exact differential. (Compare Example 12 below). The determination of the circumstances under which such integrable combinations can be made in the general case offers considerable difficulties when attacked directly: and I have failed to make this determination.

36. The following remark relating to the case in which (29) has three unequal roots may be of interest. Since H, K, L are zero, we have

$$\left\| \begin{matrix} \frac{du_1}{dr}, & \frac{du_1}{ds}, & \frac{du_1}{dt} \\ \frac{du_2}{dr}, & \frac{du_2}{ds}, & \frac{du_2}{dt} \end{matrix} \right\| = 0,$$

where u_1, u_2 are arguments of one first integral.

Now these equations may mean either of two things; viz., that u_1, u_2 contain r, s, t in the same combination; or that one of them does not contain r, s, t at all.

In the first case, we can eliminate r, s, t from the two equations

$$u_1 = c_1, \quad u_2 = c_2,$$

obtaining a relation between p, q, z, x, y, c_1, c_2. The same may be done with each of the three first integrals. From the three results thus obtained we may eliminate p, q, leaving a complete primitive involving six constants. It may happen that the final result is simply an identity; but, if so, it would probably throw some light on the question proposed in paragraph 35.

In the other case, in which u_2 does not involve r, s, t, the equation $u_2 = \text{const.}$ is a relation between p, q, z, x, y and one constant. Com-

bining this with other relations similarly obtained from the two other first integrals, we may get a complete primitive, or an identity, involving three, four, or five arbitrary constants.

Class III. *Linear Equations—(continued).*

(2). $A = D = 0$.

37. Here we need only consider the case in which B does not vanish; and replacing that coefficient by unity, the equation to be discussed is

$$\beta - C\gamma - M = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38).$$

Now if we try to get the equations for determining the first integrals of this equation, we shall find that different sets are obtained according as we assume that b or b' does not vanish. These cases may be discussed separately.

38. *Case in which b does not vanish.*—The equations (27) give at once

$$b' = 0, \quad c = 0, \quad c' = C.$$

The equations (7) proper for this case are (i.), (iv.), (v.), which become (since $b = B = 1$),

$$\left.\begin{aligned} \left(\frac{du}{dy}\right) + M\frac{du}{dr} &= 0 \\ \frac{du}{ds} + C\frac{du}{dr} &= 0 \\ \frac{du}{dt} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (39).$$

But the values of M, C derived from these equations, reduce (38) to the form

$$\frac{du}{dr}\beta + \frac{du}{ds}\gamma + \left(\frac{du}{dy}\right) = 0, \quad \frac{du}{dt} = 0;$$

so that (38) can be made an exact differential by a factor. Its first integral is found in the form $u = \phi(x)$,

u being a certain definite function of r, s, p, q, z, x, y.

It may be noticed that the equations satisfied by common solutions of (39) are

$$\left.\begin{aligned} \frac{dC}{dt}\cdot\frac{du}{dr} &= 0 \\ \frac{du}{dq} + \frac{dM}{dt}\cdot\frac{du}{dr} &= 0 \\ \frac{du}{dp} + \left\{\frac{dM}{ds} + C\frac{dM}{dr} - \left(\frac{dC}{dy}\right) - M\frac{dC}{dr}\right\}\frac{du}{dr} &= 0 \end{aligned}\right\} \quad \ldots\ldots (40).$$

Further, by combining the first of (39) with the second of (40), we get an independent equation

$$\frac{du}{dz} + \&c. = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (41).$$

It appears then, that, in order that (38) may have a first integral, (on

the supposition that b does not vanish,) it is necessary that

$$\frac{d\mathrm{C}}{dt} = 0.$$

Even when this condition is fulfilled we cannot find a second integral. To get the first integral in this case we may either employ the systems (39), (40), and (41); or, what is usually easier, seek a factor which will render (38) an exact differential.

39. *Case in which b' does not vanish.*—In this case we learn from (27) that b must vanish, and that either c or c' must vanish. Take first the case in which c' vanishes.

The equations (7) for finding the first integral reduce to (ii.), (vii.) (ix.); viz., these are

$$\left(\frac{du}{dx}\right) + \mathrm{M}\frac{du}{ds} = 0,$$
$$\frac{du}{dr} = 0,$$
$$\frac{du}{dt} + \mathrm{C}\frac{du}{ds} = 0.$$

Putting the values of M, C in (38), and remembering that u does not involve r, it is seen that, on being multiplied by a certain factor, viz. $\frac{du}{ds}$, the equation becomes an exact differential; viz.,

$$\frac{du}{ds}\beta + \frac{du}{dt}\gamma + \left(\frac{du}{dx}\right) = 0.$$

The first integral is of the form $u = \phi(y)$.

As in the last case, we are unable to carry the integration further.

40. We have now left the case in which

$$b = 0, \quad b' = 1, \quad c = 0, \quad c' = \mathrm{C}.$$

The equations for the first integral are

$$\left.\begin{aligned} \left(\frac{du}{dx}\right) - \mathrm{C}\left(\frac{du}{dy}\right) + \mathrm{M}\frac{du}{ds} &= 0 \\ \frac{du}{dr} &= 0 \\ \frac{du}{dt} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (42).$$

To which are to be added

$$\left.\begin{aligned} \frac{du}{dp} - \frac{d\mathrm{C}}{dr}\left(\frac{du}{dy}\right) + \frac{d\mathrm{M}}{dr}\cdot\frac{du}{ds} &= 0 \\ \mathrm{C}\frac{du}{dq} + \frac{d\mathrm{C}}{dt}\left(\frac{du}{dy}\right) - \frac{d\mathrm{M}}{dt}\cdot\frac{du}{ds} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (43).$$

From these equations we learn that s is the only coefficient of the

second order which occurs in the first integral. It is to be noted also that the second equation of (43) cannot become an identity because the case in which C vanishes would require that c' should vanish; or, it would be the case of the last paragraph.

Another equation may be formed from the second of (42) and the first of (43); viz., $\frac{d^2C}{dr^2}\left(\frac{du}{dy}\right) - \frac{d^2M}{dr^2}\cdot\frac{du}{ds} = 0.$

Now, if this be not an identity, there can be no first integral; for by its help we can reduce the first of (42) to the form

$$\frac{du}{dx} + p\frac{du}{dz} + S_1\frac{du}{ds} = 0,$$

and the first of (43) to the form

$$\frac{du}{dp} + S_2\frac{du}{ds} = 0.$$

The combination of these would give a seventh equation of the form

$$\frac{du}{dz} + S_3\frac{du}{ds} = 0.$$

Hence two necessary conditions for a first integral are that

$$\frac{d^2C}{dr^2} = 0, \quad \frac{d^2M}{dr^2} = 0.$$

By a similar process, or from symmetrical considerations, we may infer

$$\frac{d^2}{dt^2}\left(\frac{1}{C}\right) = 0, \quad \frac{d^2}{dt^2}\left(\frac{M}{C}\right) = 0$$

are also necessary conditions.

41. *Example* 10.

$$(xs+yt)\beta - (xr+ys)\gamma + \left(\frac{x}{q} - \frac{y}{p}\right)(s^2-rt)s = 0 \quad \ldots\ldots \text{(i.)}$$

The equations for the first integral are

$$(xs+yt)\left(\frac{du}{dx}\right) - (xr+ys)\left(\frac{du}{dy}\right) - \left(\frac{x}{q} - \frac{y}{p}\right)s(s^2-rt)\frac{du}{ds} = 0 \ldots \text{(ii.)},$$

$$\frac{du}{dr} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii.)},$$

$$\frac{du}{dt} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv.)}$$

The derived system is

$$x\left(\frac{du}{dy} + q\frac{du}{dz} + t\frac{du}{dq}\right) - yt\frac{du}{dp} - \left(\frac{x}{q} - \frac{y}{p}\right)st\frac{du}{ds} = 0,$$

$$y\left(\frac{du}{dx} + p\frac{du}{dz} + r\frac{du}{dp}\right) - xr\frac{du}{dq} + \left(\frac{x}{q} - \frac{y}{p}\right)rs\frac{du}{ds} = 0.$$

The elimination of $\frac{du}{ds}$ between these gives

$$yt\frac{du}{dx} + xr\frac{du}{dy} + (ypt+xqr)\frac{du}{dz} = 0.$$

Combining this with (iii.), (iv.) in turn,

$$\frac{du}{dx}+p\frac{du}{dz}=0,\quad \frac{du}{dy}+q\frac{du}{dz}=0.$$

The other equations reduce to (iii.), (iv.), and

$$y\frac{du}{dp}-x\frac{du}{dq}+\left(\frac{x}{q}-\frac{y}{p}\right)s\frac{du}{ds}=0.$$

The ordinary equations formed from this system are

$$pq\,yds-(xp-yq)\,sdp=0,$$
$$ydq+xdp=0,$$
$$dz-pdx-qdy=0.$$

The first two give $pqds+s\,(pdq+qdp)=0$, or $u_1=pqs$.

The last pair give

$$dz-(pdx+xdp)-(qdy+ydq)=0,\quad \text{or}\quad u_2=z-px-qy.$$

Thus the first integral of (i.) is

$$\mathrm{F}\{spq,\ z-xp-yq\}=0.$$

42. *Example* 11.

$$(qs-pt)\,\beta-(qr-ps)\,\gamma+s\,(s^2-rt)=0 \qquad \text{(i.)}$$

The equations for the first integral are

$$(qs-pt)\left(\frac{du}{dx}\right)-(qr-ps)\left(\frac{du}{dy}\right)-s\,(s^2-rt)\frac{du}{ds}=0 \qquad \text{(ii.)},$$

$$\frac{du}{dr}=0,\quad \frac{du}{dt}=0,$$

$$\left.\begin{aligned} p\left(\frac{du}{dx}\right)+(qr-ps)\frac{du}{dq}-sr\frac{du}{ds}&=0\\ q\left(\frac{du}{dy}\right)-(qs-pt)\frac{du}{dp}-st\frac{du}{ds}&=0\end{aligned}\right\} \qquad \text{(iii.)}$$

The equations (ii.), (iii.) are easily reduced to the forms

$$q\frac{du}{dq}+p\frac{du}{dp}-s\frac{du}{ds}=0,$$
$$\frac{du}{dx}+p\frac{du}{dz}=0,\quad \frac{du}{dy}+q\frac{du}{dz}=0.$$

The first of this trio gives the required first integral

$$u=\mathrm{F}\{sp,\ sq\}=0.$$

43. *Example* 12. As a last example we take an equation which has three first integrals, viz., $\beta-s\gamma=0$ ..... (i.)

Treating this as in paragraphs 38 and 39, we get two first integrals

$$\mathrm{F}\{2r-s^2,\ x\}=0 \qquad \text{(ii.)},$$
$$\mathrm{F}\{\log s-t,\ y\}=0 \qquad \text{(iii.)}$$

Again, throwing it into the form

$$\frac{ds}{dx}-s\frac{ds}{dy}=0,$$

(which is equivalent to the process of paragraph 40,) we find

$$F\{xs+y,\ s\} = 0 \quad \ldots\ldots\ldots\ldots\ldots \text{(iv.)}$$

By a combination of these we can get two second integrals. Say we take from (iv.) $s = c$,

from (ii.), (iii.) we have $r = \frac{1}{2}c^2 + \phi(x)$,

$$t = \log c + \psi(y).$$

Substituting in $dp = rdx + sdy$, $dq = sdx + tdy$, and integrating,

$$p = \tfrac{1}{2}c^2 x + \phi_1(x) + cy + \varphi_2(c) \quad \ldots\ldots\ldots\ldots \text{(v.)},$$

which may be generalized by taking c to be a function of x, y defined by the equation

$$0 = cx + y + \phi_2'(c) \quad \ldots\ldots\ldots\ldots \text{(vi.)},$$

and

$$q = cx + \log c \,.\, y + \psi_1(y) + \psi_2(c) \quad \ldots\ldots\ldots \text{(vii.)},$$

where c may be considered to be defined by

$$0 = x + \frac{y}{c} + \psi_2'(c) \quad \ldots\ldots\ldots\ldots \text{(viii.)}$$

Again, the values of (v.), (vii.) being such that

$$\frac{dp}{dy} = \frac{dq}{dx},$$

the expression $dz - pdx - qdy$ is an exact differential; and we have, for a complete primitive of (i.),

$$z = \tfrac{1}{4}c^2x^2 + \Phi(x) + \phi_2(c) \,.\, x + cxy + \tfrac{1}{2}\log c \,.\, y^2 + \Psi(y) + \psi_2(c) \,.\, y + \chi(c) \ldots\ldots \text{(ix.)}$$

The general solution is found by eliminating c between (ix.) and its derivative with respect to c; viz.,

$$0 = \tfrac{1}{2}cx^2 + \phi_2'(c) \,.\, x + xy + \frac{1}{2c}y^2 + \psi_2'(c) \,.\, y + \chi'(c).$$

This reduces, on account of (vi.), (viii.), to

$$0 = \tfrac{1}{2}cx^2 + xy + \frac{y^2}{2c} - \chi'(c) = 0 \quad \ldots\ldots\ldots\ldots \text{(x.)}$$

The required solution therefore consists of the four equations (vi.), (viii.), (ix.), (x.).

It is easy to reduce this to a less complicated form. The equations (vi.), (viii.), (x.) may be written

$$\begin{aligned} cx + y &= -\phi_2'(c) \\ &= -c\psi_2'(c) \\ &= \sqrt{2c\chi'(c)}. \end{aligned}$$

The three functions ϕ_2, ψ_2, χ are therefore not independent, and can be expressed in terms of any one of them. At the same time the three equations (vi.), (viii.), (x.) reduce to one only. For instance, the general solution of (i.) may be put in the form

$$z = \Phi(x) + \Psi(y) + \tfrac{1}{4}c^2x^2 + cxy + \tfrac{1}{2}\log c \,.\, y^2 + x \,.\, \phi(c) + y\int \phi'(c)\frac{dc}{c} + \int \{\phi'(c)\}^2 \frac{dc}{2c} \ldots\ldots \text{(xi.)},$$

where c is defined by the equation

$$0 = \phi'(c) + cx + y \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(xii.)}$$

It is perhaps scarcely necessary to add that this form of solution cannot be further simplified by substituting $cx+y$ for $-\phi'(c)$ under the signs of integration; for in the integral

$$-\int (cx+y)\, dc,$$

for instance, x, y are tacitly regarded as independent of c; and this assumption is erroneous.

May 10*th*, 1877.

Lord RAYLEIGH, F.R.S., President, in the Chair.

Mr. C. Pendlebury, B.A., was admitted into the Society.

The following communications were made to the Society:—

"Note on the Correlation of two Planes:" Dr. Hirst (abstract read by Mr. Tucker).

"The Irreducible Case:" Herr Weichold, of the Johanneum, Zittau, Saxony (abstract given).

(1) "Generalization of Cases of Five-bar Motion considered at the April meeting;" (2) "The Kinematic Paradox;" (3) "A Method of Solving by Linkwork $f(x) = 0$, an algebraical equation of the n^{th} degree:" Mr. Harry Hart. (Mr. Kempe spoke upon these papers, clearing up some parts of the investigation left unsolved by the Author; Prof. Cayley also spoke on the same subject.)

"On the Free Motion of a Solid through an Infinite Mass of Perfect Liquid:" Prof. Lamb, Melbourne University.

Mr. J. W. L Glaisher stated that he had had all the cases in which there were more than 50 consecutive composite numbers looked out from Burckhardt's and Dase's factor tables, which extend from 1 to 3,000,000, and from 6,000,000 to 9,000,000. A list of the groups of composite numbers so found in the first three millions was exhibited to the Society. The two most noteworthy stretches were a group of 111 consecutive numbers between 370,261 and 370,373, and of 113 between 492,113 and 492,227. These two very long sets of numbers without a prime, occurring in the first half-million, were remarkable, as he thought they exceeded what one would *à priori* expect. The longest stretch noted in the three millions occurred in the third million, and consisted of 147 consecutive composite numbers; and the next longest occurred in the second million, where there were two stretches each of 131.

Profs. Cayley and Clifford put each a question to the members present asking for information.

The following presents were received :—

Carte de Visite likeness of Mr. A. B. Kempe.

"La théorie Hugodécimale ou la base scientifique et définitive de l'Arithmo-Logistique universelle," par le Comte Léopold Hugo (Paris, 1877) : from the Author.

Note on the Correlation of Two Planes. *By* T. A. Hirst, F.R.S.

[*Read May 10th*, 1877.]

Introduction.

65. In a former paper* on this subject, published in the Proceedings of the London Mathematical Society (Vol. v., p. 40), the nature and properties were described—*first*, of an ordinary correlation satisfying any eight given conditions ; *secondly*, of an exceptional correlation of the first order, possessing either a singular point or a singular line in each plane, and satisfying seven conditions ; and, *thirdly*, of an exceptional correlation of the second order, having in each plane not only a singular point, but also a singular line passing through that point, and satisfying six conditions. Moreover, the two following numerical relations were established (art. 20) between the π and the λ exceptional correlations of the first order, with singular points and singular lines, respectively, which satisfy any seven conditions, and the μ and the ν ordinary correlations which, besides satisfying these same conditions, possess a given pair of conjugate points and of conjugate lines, respectively :

$$2\nu = \mu + \pi,$$
$$2\mu = \nu + \lambda.$$

It was by means of these relations, in fact, that the number of ordinary correlations was determined which satisfy any eight elementary conditions (art. 22). Before they could be applied, however, the exceptional correlations of the first order which satisfy any seven elementary conditions had to be directly determined, and this determination not unfrequently necessitated the consideration of the projective properties of curves of high order.

In the present paper I propose to show that the object just referred to can be attained in a very much simpler manner by means of two numerical relations, hitherto unobserved, connecting the exceptional

* In order that the articles of my former paper may be conveniently cited, those of the present one bear the next succeeding numbers.

correlations of the second order which satisfy six conditions, with the exceptional correlations of the first order which, besides satisfying the six conditions in question, possess a given pair either of conjugate points or conjugate lines.

Correlations satisfying six conditions.

66. The correlations which satisfy six conditions obviously form a two-fold system. They consist, in general, of a doubly infinite number of ordinary correlations, a singly infinite number of exceptional correlations of the first order, and a finite number θ of exceptional correlations of the second order.*

The exceptional correlations of the first order will, in general, arrange themselves into two one-fold systems; those of the one system having singular points (*centres*), whilst those of the other have singular lines (*axes*). They may be more conveniently distinguished as *central* correlations and *axial* correlations, respectively. In each plane the centres of the former will, in general, have a certain locus, and the axes of the latter a certain envelope. Moreover, between the points of the two *loci of centres*, as well as between the tangents of the two *envelopes of axes*, there will be a (1, 1) correspondence, corresponding elements being, in the one case, associated centres, and, in the other, associated axes.†

An exceptional correlation of the second order presents itself whenever the associated centres of a central correlation lie on the associated axes of an axial one. Such exceptional correlations of the second order, in fact, may be appropriately termed *central and axial*, and they may be regarded as the connecting links between the two systems of exceptional correlations of the first order above alluded to.

67. In the one-fold system of central correlations which satisfy six conditions there will be a certain number $\dot{\pi}$ for which two given points are conjugate, and a certain number $\bar{\pi}$ for which two given lines are conjugate. In like manner, the one-fold system of axial correlations which satisfy the same six conditions will include a certain number $\dot{\lambda}$ for which two given points are conjugate, and another number $\bar{\lambda}$ for which two given lines are conjugate. This being the case, the two new relations, referred to in art. 65, may be thus expressed:

$$2\dot{\pi} = \bar{\pi} + \theta,$$
$$2\bar{\lambda} = \dot{\lambda} + \theta,$$

* I do not consider cases where, in consequence of the peculiar nature of the six conditions, no such distribution of correlations prevails,—where, for instance θ is infinite.

† The further consideration of these loci and envelopes, being foreign to the purpose of the present note, is omitted. Their determination, when the conditions are of an elementary character, presents, however, no difficulty whatever.

where θ denotes the number of central and axial correlations which satisfy the six conditions under consideration.

Since each of these equations is deducible from the other by the Principle of Duality, a demonstration of the first one will clearly suffice. To this end let a_1 and a_2 denote any two lines whatever in the first and second planes respectively, and A_2', A_2'' any two points whatever in a_2. A point A_1' being taken arbitrarily in a_1, we have by hypothesis $\dot{\pi}$ central correlations in which A_1' and A_2' are conjugate points; and since in each of these correlations A_2'' will also have a conjugate point A_1'' in a_1, we may say that to each point A_1' in a_1 correspond $\dot{\pi}$ points A_1'', and in like manner that to each point A_1'' correspond $\dot{\pi}$ points A_1'. By a well-known theorem, therefore, there will, in general, be $2\dot{\pi}$ points in a_1 with each of which two corresponding points A_1', A_1'' will coincide.

The circumstances under which such coincidences take place can be readily assigned. One will present itself in each of the θ central and axial correlations which satisfy the six conditions under consideration; for A_2' and A_2'' being arbitrary points, their corresponding lines in any such correlation will coincide with the axis thereof (art. 16, third case). The only other central correlations, however, in which such a coincidence can occur are those which have a centre either in a_1 or in a_2; the lines corresponding to A_2' and A_2'' will in the former case both pass through the centre in a_1, whilst in the latter case these will coincide with one another (art. 16, first case).

But the correlations which have a centre in a_1, together with those which have a centre in a_2, precisely make up the total number $\bar{\pi}$ of correlations in which a_1 and a_2 are conjugate lines; for the point which corresponds to a_1 in a correlation whose centre Σ_1 lies in a_1, being an indeterminate point of a line passing through the associated centre Σ_2 (art. 16, first case), may be regarded as situated in a_2; whilst in any correlation whose centre Σ_1 is not in a_1, the associated centre Σ_2 will itself correspond to a_1; so that this line will be conjugate to a_2 when, and only when, Σ_2 falls in the latter line.*

The first of the above equations, therefore, is simply an expression of the fact that the $2\dot{\pi}$ coincidences of A_1', and A_1'' are, individually, produced by the $\theta+\bar{\pi}$ correlations above specified.

Central and Axial Correlations satisfying six elementary conditions.

68. In order to apply the two equations just established to the problem referred to in art. 65, the central and axial correlations which satisfy any six elementary conditions must, of course, be first directly

* From this it follows that $\bar{\pi}$ is equal to the sum of the orders of the two *loci of centres* (art. 66).

determined. It doing so, it will be assumed, as in art. 27, that no special relations of position exist between the given points and lines which these conditions involve.

If any combination of such conditions be represented by its signature $(\alpha\beta\gamma\delta)$, where, as before (art. 23), α denotes the number of given points in the first plane which correspond, individually, to given lines in the second, β the number of lines in the first plane which correspond to given points in the second, γ the number of given pairs of conjugate points, δ the number of given pairs of conjugate lines $(2\alpha+2\beta+\gamma+\delta=6)$; and if, moreover, it be remembered that $(\alpha\beta\gamma\delta)$ and $(\beta\alpha\gamma\delta)$ represent essentially the same conditions, it will be readily seen that the total number of cases to be considered amounts to twenty. From the Principle of Duality, however, it is manifest that the values of θ must be the same for the two signatures $(\alpha\beta\gamma\delta)$ and $(\alpha\beta\delta\gamma)$; so that we may confine our attention to the thirteen cases indicated in the twelve next following articles.

The process of reasoning which leads to the determination of θ is in all cases very simple, being based entirely upon the following three obvious *properties of a central and axial correlation* (art. 16, third case):

a. If a point and line correspond to one another, one of the associated centres will either coincide with the point or lie on the line, and one of the associated axes will either coincide with the line or pass through the point.

b. If two lines be conjugate to each other, one of the associated centres will lie on one of the lines.

c. If two points be conjugate to each other, one of the associated axes will pass through one of the points.

69. The signature being (3000), let the data be denoted by (3000

$$\begin{vmatrix} P_1 & Q_1 & R_1 \\ p_2 & q_2 & r_2 \end{vmatrix},$$

and the associated centres and axes of any central and axial correlation $\theta=6$ satisfying the conditions by Σ_1, σ_1 and Σ_2, σ_2.

Since one, at least, of the three points $P_1 Q_1 R_1$ will not lie in σ_1, it follows from art. 68, **a**, that σ_2 must coincide with one of the three lines $p_2 q_2 r_2$, say p_2, and that the axis σ_1 associated therewith must pass through the two non-corresponding points $Q_1 R_1$. Moreover, by the Principle of Duality, the centre Σ_1 in σ_1 must coincide with one of these two points, say Q_1, and its associate Σ_2 with the intersection of the two non-corresponding lines $r_1 p_1$. From this we at once conclude that there are but six different distributions of the associated centres

and axes admissible. On the other hand, for each distribution, such as

Σ_1	σ_1	Σ_2	σ_2
Q_1	$\overline{Q_1 R_1}$	$(p_2 r_2)$	p_2

one central and axial correlation, and only one, satisfying the given six conditions, is clearly possible. Hence $\theta = 6$.

(2100) 70. In the next case, the signature of the six conditions is (2100), the corresponding data being

$$\left| \begin{matrix} P_1 & Q_1 & r_1 \\ p_2 & q_2 & R_2 \end{matrix} \right| .$$

$\theta=1$ Since one, at least, of the lines $p_2 q_2$ will not coincide with the axis σ_2, its associate σ_1 will necessarily pass through one, at least, of the points $P_1 Q_1$, and therefore not coincide with r_1. But under these circumstances σ_2 will necessarily pass through R_2, and therefore coincide neither with p_2 nor q_2; whence we conclude that σ_1 must pass through both the points $P_1 Q_1$. Correlatively, the centre Σ_1 must lie in r_1, and Σ_2 must coincide with the intersection of p_2 and q_2. The only admissible distribution of the associated centres and axes, therefore, is

Σ_1	σ_1	Σ_2	σ_2
$(\overline{P_1 Q_1}, r_1)$	$\overline{P_1 Q_1}$	$(p_2 q_2)$	$\overline{(p_2 q_2) R_2}$

With this arrangement, however, a central and axial correlation satisfying the conditions is clearly possible; hence $\theta = 1$.

(2020) 71. In the two cases above treated the conditions were all double ones; in all the remaining cases an even number of single conditions are involved, and the properties **b** and **c** of art. 68 have to be borne in mind. The first of such cases has the signature (2020), the data being

$$\left| \begin{matrix} P_1 & Q_1 & A_1 & B_1 \\ p_2 & q_2 & A_2 & B_2 \end{matrix} \right| .$$

$\theta=2$ Here, as in the last case, we know that σ_1 must pass through one, at least, of the two points $P_1 Q_1$, and hence that one, at least, of the two points $A_1 B_1$ will certainly not lie on σ_1; but if so, σ_2 will necessarily pass through one, at least, of the points $A_2 B_2$, and accordingly coincide with neither p_2 nor q_2. Hence σ_1 will necessarily pass through both P_1 and Q_1, and its associate σ_2 through A_2 and B_2. Moreover, as in art. 69, Σ_1 must coincide with one or other of the two points $P_1 Q_1$, and its associate on $\overline{A_2 B_2}$ must lie on the non-corresponding line q_2 or p_2. The only two admissible distributions, therefore, are

Σ_1	σ_1	Σ_2	σ_2
P_1	$\overline{P_1 Q_1}$	$(q_2, \overline{A_2 B_2})$	$\overline{A_2 B_2}$
Q_1	$\overline{P_1 Q_1}$	$(p_2, \overline{A_2 B_2})$	$\overline{A_2 B_2}$

and since a central and axial correlation is possible with each, we have $\theta = 2$.

72. The signature in the next case is (2011), the data being (2011)

$$\left| \begin{matrix} P_1 & Q_1 & A_1 & b_1 \\ p_2 & q_2 & A_2 & b_2 \end{matrix} \right| .$$

As before, σ_1 must pass through one, at least, of the points $P_1 Q_1$; but $\theta=7$ here it may, or may not, pass also through A_1. If it do not, its associate σ_2 will pass through A_2, and, as in the last case, σ_1 will pass through both points $P_1 Q_1$. If, on the other hand, σ_1 pass through A_1, and therefore through one only of the points $P_1 Q_1$, σ_2 will necessarily coincide with the non-corresponding line q_2 or p_2. From the Principle of Duality we infer at once that the centre Σ_2 must either coincide with the intersection of p_2 with q_2, or with that of one of these lines and b_2. This being the case, it is obvious that the associated centres and axes must be distributed in one of the following seven ways:

Σ_1	σ_1	Σ_2	σ_2
$(\overline{P_1 Q_1}, b_1)$	$\overline{P_1 Q_1}$	$(p_2 q_2)$	$\overline{A_2 (p_2 q_2)}$
P_1	$\overline{P_1 Q_1}$	$(b_2 q_2)$	$\overline{A_2 (b_2 q_2)}$
Q_1	$\overline{P_1 Q_1}$	$(p_2 b_2)$	$\overline{A_2 (p_2 b_2)}$
$(\overline{A_1 Q_1}, b_1)$	$\overline{A_1 Q_1}$	$(p_2 q_2)$	p_2
Q_1	$\overline{A_1 Q_1}$	$(p_2 b_2)$	p_2
$(\overline{P_1 A_1}, b_1)$	$\overline{P_1 A_1}$	$(p_2 q_2)$	q_2
P_1	$\overline{P_1 A_1}$	$(b_2 q_2)$	q_2

Since each distribution determines, as before, one, and only one, central and axial correlation satisfying the given conditions, we have in the present case $\theta = 7$.

73. Passing next to the case (1120), where the data are (1120)

$$\left| \begin{matrix} P_1 & q_1 & A_1 & B_1 \\ p_2 & Q_2 & A_2 & B_2 \end{matrix} \right| ,$$

I observe that σ_2 cannot coincide with p_2, because σ_1 would then coin- $\theta=4$ cide with q_1, whereas one of the associated axes ought to pass through one of the conjugate points $A_1 A_2$ (art. 68, **c**). Hence σ_1 must pass through P_1 as well as through one of the points $A_1 B_1$, and its associate σ_2 through Q_2 as well as through one of the points $B_2 A_2$. Moreover the centre Σ_1 must either coincide with P_1 or lie in q_1, and its associate Σ_2 must, in the former case, coincide with Q_2, and in the latter lie in p_2. In short, the following four distributions of associated centres and axes are the only admissible ones; accordingly $\theta = 4$.

Σ_1	σ_1	Σ_2	σ_2
P_1	$\overline{P_1 A_1}$	Q_2	$\overline{Q_2 B_2}$
$(\overline{P_1 A_1}, q_1)$	$\overline{P_1 A_1}$	$(\overline{Q_2 B_2}, p_2)$	$\overline{Q_2 B_2}$
P_1	$\overline{P_1 B_1}$	Q_2	$\overline{Q_2 A_2}$
$(\overline{P_1 B_1}, q_1)$	$\overline{P_1 B_1}$	$(\overline{Q_2 A_2}, p_2)$	$\overline{Q_2 A_2}$

(1111) 74. In the next case (1111), where the data

$$\left| \begin{matrix} P_1 & q_1 & A_1 & b_1 \\ p_2 & Q_2 & A_2 & b_2 \end{matrix} \right|$$

$\theta=2$ are of a perfectly symmetrical character, it is easily seen that, although the associated axes σ_1, σ_2 must, as before, pass through P_1, Q_2, respectively, these points cannot be associated centres, since one of two such centres must lie in one of the lines $b_1 b_2$. The following two distributions of associated centres and axes, therefore, are alone admissible, and $\theta = 2$.

Σ_1	σ_1	Σ_2	σ_2
$(P_1 A_1, q_1)$	$P_1 A_1$	$(p_2 b_2)$	$Q_2 (p_2 b_2)$
$(b_1 q_1)$	$P_1 (b_1 q_1)$	$(p_2 Q_2 A_2)$	$Q_2 A_2$

(1040) 75. The six conditions whose signature is (1040) cannot, in general,
$\theta=0$ be satisfied by any central and axial correlation. For four points in the first plane could not be conjugate to four given points in the second, unless σ_1 passed through two of the former and σ_2 through the conjugates of the remaining two; but if the positions σ_1 and σ_2 were thus determined, the residual double condition, in virtue of which a given point P_1 is to correspond to a given line p_2, would clearly not be satisfied, since, in general, σ_1 would neither pass through P_1 nor σ_2 coincide with p_2 (art. 68, **a**).

(1031) 76. In the next case, however, where the signature is (1031) and the data

$$\left| \begin{matrix} P_1 & A_1 & B_1 & C_1 & d_1 \\ p_2 & A_2 & B_2 & C_2 & d_2 \end{matrix} \right|,$$

$\theta=6$ it may easily be shown that $\theta = 6$. In fact, since σ_1 cannot pass through more than two of the three points $A_1 B_1 C_1$, σ_2 must pass through one, at least, of the three points $A_2 B_2 C_2$ (art. 68, **c**). From this, however, or rather from the fact that σ_2 cannot coincide with p_2, it follows that σ_1 must pass through P_1 (art. 68, **a**), and hence that there will always be two, at least, of the three points $A_1 B_1 C_1$ through which σ_1 will not pass. Accordingly σ_2 will necessarily pass through two of the three points $A_2 B_2 C_2$, and σ_1 through P_1 and the conjugate of the third. But in each of the three positions of σ_1, thus proved to be admissible, Σ_1 may, and must, either coincide with P_1 or lie in d_1, whilst in the associated axis σ_2, Σ_2 will, in the one case, lie on d_2, and in the other on p_2. The six solutions require no further definition in the present case.

(1022) 77. When the signature is (1022), the data being

$$\left| \begin{matrix} P_1 & A_1 & B_1 & c_1 & d_1 \\ p_2 & A_2 & B_2 & c_2 & d_2 \end{matrix} \right|,$$

σ_2 may coincide with p_2, provided σ_1 pass through the two points A_1B_1. $\theta=9$ There are, in fact, two, and only two, solutions of this kind, since Σ_1 may, and must, lie either on c_1 or on d_1, and Σ_2 either on d_2 or c_2, respectively. In all other cases σ_1 must pass through P_1, and σ_2 through one, at least, of the points A_2B_2. From the Principle of Duality, applied to the preceding reasoning, we at once infer that in two of these cases Σ_1 will coincide with P_1 and Σ_2 with (c_2d_2); in all the remaining ones, however, Σ_2 will lie in p_2, and Σ_1 in one, at least, of the lines c_1d_1. If Σ_2, indeed, do not coincide with either of the intersections of p_2 with c_2 and d_2, Σ_1 will lie in both the lines c_1d_1, and σ_2 will pass through both the points A_2B_2. But if, on the other hand, Σ_2 coincide either with (c_2p_2) or (d_2p_2), σ_2 passing of course through A_2 or B_2, σ_1 will pass, respectively, through B_1 or A_1, and Σ_1 will lie in d_1 or c_1; there are clearly four solutions of this kind. For the sake of greater clearness the nine distributions of associated centres and axes above indicated are collected in the following table:

Σ_1	σ_1	Σ_2	σ_2
$(c_1, \overline{A_1B_1})$	$\overline{A_1B_1}$	(d_2p_2)	p_2
$(d_1, \overline{A_1B_1})$	$\overline{A_1B_1}$	(c_2p_2)	p_2
P_1	$\overline{A_1P_1}$	(c_2d_2)	$\overline{B_2(c_2d_2)}$
P_1	$\overline{B_1P_1}$	(c_2d_2)	$\overline{A_2(c_2d_2)}$
(c_1d_1)	$\overline{(c_1d_1)P_1}$	$(p_2, \overline{A_2B_2})$	$\overline{A_2B_2}$
$(d_1, \overline{B_1P_1})$	$\overline{B_1P_1}$	(c_2p_2)	$\overline{A_2(c_2p_2)}$
$(c_1, \overline{B_1P_1})$	$\overline{B_1P_1}$	(d_2p_2)	$\overline{A_2(d_2p_2)}$
$(d_1, \overline{A_1P_1})$	$\overline{A_1P_1}$	(c_2p_2)	$\overline{B_2(c_2p_2)}$
$(c_1, \overline{A_1P_1})$	$\overline{A_1P_1}$	(d_2p_2)	$\overline{B_2(d_2p_2)}$

It is scarcely necessary to add that one, and only one, central and axial correlation satisfying the required conditions exists with each of the above distributions; so that we have $\theta = 9$.

78. The only combinations of six conditions which remain to be treated are those in which these conditions are all single ones. In the (0060) two cases (0060) and (0051), where more than four pairs of conjugate (0051) points are given, the six conditions, clearly, cannot be satisfied by any $\theta=0$ axial correlation whatever (see arts. 34 and 75).

79. In the case whose signature is (0042) it is easy to see that $\theta=12$. (0042) For σ_1 must be one of the six lines passing through two of the four $\theta=12$ given points in the first plane, and Σ_1 must lie in one or other of the two given lines in that plane; the axis σ_2 associated with the former will, of course, pass through the conjugates of the remaining two points,

and the centre Σ_2, associated with the latter, will lie in the conjugate of the remaining line.

80. Finally, in the case where the signature is (0033), and the symmetry between the two planes is perfect, it is just as easy to see that there will be eighteen solutions, or $\theta = 18$. In nine of these σ_1 will pass through two of the given points of the first plane, and σ_2 through the conjugate of the third as well as through the intersection of two of the three given lines in the second plane; with this intersection, moreover, the centre Σ_2 will coincide, whilst its associate Σ_1 will lie on the conjugate of the line which does not pass through Σ_2. In the nine remaining solutions the distribution of associated centres and axes is of the same character, the two planes being merely interchanged. (0033) $\theta=18$

Axial (or Central) Correlations satisfying seven elementary conditions.

81. The values of θ for any six elementary conditions being now known, we have merely to show how, by means of the two equations of art. 67, the number of exceptional correlations of the first order (central or axial) may be determined which satisfy any seven such conditions.

The number λ of axial correlations, for instance, which satisfy the seven conditions $(\alpha\beta\gamma\delta)$ is, by definition, identical with the value of $\dot{\lambda}$ answering to the six conditions $(\alpha\beta\,\overline{\gamma-1}\,\delta)$ (art. 67). If this number were known, as well as the value of θ for these six conditions, the second equation in art. 67 would at once give the value of $\overline{\lambda}$ for these conditions, or, in other words, the value of λ for the seven conditions $(\alpha\beta\,\overline{\gamma-1}\,\overline{\delta+1})$. The values of λ, therefore, answering to the several combinations $(\alpha\beta\gamma\delta)$ of a group of seven conditions in which α and β retain their values, may be successively deduced from the value of λ answering to any one of them;*—for instance, from the value of λ for the signature $(\alpha\beta\ 7-\overline{2\alpha}-\overline{2\beta}\ 0)$ in which no conjugate lines are involved.

From this it follows at once that the values of λ for all possible cases of seven elementary conditions may be deduced from the values of λ which correspond to the following six combinations of such conditions:

(3010), (2110), (2030), (1130), (1050), (0070).

But these values may be very readily determined directly; in fact, they have already been found to be

0, 1, 0, 0, 0, 0,

respectively. (See arts. 28, 29, 30, 32, 34, 37.)

* A similar conclusion will obviously hold if, in place of the above elementary double conditions, any conditions, which are invariable for one and the same group, are substituted.

82. The number π of central correlations which satisfy any seven elementary conditions $(\alpha\beta\gamma\delta)$ may be found in a precisely similar manner. Since this number, however, by the Principle of Duality, is the same as the number λ of axial correlations which satisfy the conditions $(\alpha\beta\delta\gamma)$, it will suffice here to tabulate the numbers given, in the several cases, by the second equation of art. 67. The grouping of these cases corresponds to that adopted in my former paper. For each group the value of $\overline{\lambda}$ occurring on any line is, for obvious reasons, the same as the value of $\dot{\lambda}$ on the next succeeding line; the values of $\dot{\lambda}$ in the first lines of the several groups correspond, of course, to the special values of λ given in the last art. The values of θ include those which have been directly determined in arts. 69—80; the latter have been supplemented, however, by means of the Principle of Duality.

Group.	$(\alpha\beta\gamma\delta)$	θ	$\dot{\lambda}$	$\overline{\lambda}$
I.	(3000)	6	0	3
II.	(2100)	1	1	1
III.	(2020)	2	0	1
	(2011)	7	1	4
	(2002)	2	4	3
IV.	(1120)	4	0	2
	(1111)	2	2	2
	(1102)	4	2	3
V.	(1040)	0	0	0
	(1031)	6	0	3
	(1022)	9	3	6
	(1013)	6	6	6
	(1004)	0	6	3
VI.	(0060)	0	0	0
	(0051)	0	0	0
	(0042)	12	0	6
	(0033)	18	6	12
	(0024)	12	12	12
	(0015)	0	12	6
	(0006)	0	6	3

Ordinary Correlations satisfying eight elementary conditions.

83. The numbers in the last column of the above table, together with those in the last column but one which stand at the head of each group, are precisely the values of λ tabulated in art. 41. These same numbers, taken, of course, in inverted order for each group, likewise coincide with

the values of π given in the table just referred to. They provide all the data, therefore, which are required for the determination, by means of the two formulæ of art. 65, of the number of ordinary correlations which satisfy any eight elementary conditions.* (See art. 42.)

With respect to each of the three exceptional cases (4000), (3100), (2200), where, the conditions being all double ones, the above mode of treatment is inapplicable, I may observe that, although the number of solutions is well known to be 1, 1, 0, respectively (see art. 43), this fact may be verified by means of either of the two general theorems given in the next article. By their aid, in fact, we may always pass from the case of two single conditions to one double one, or *vice versâ*,—a passage which, especially when the conditions are no longer elementary ones, may often be made advantageously.

84. The theorems in question may be thus enunciated:

Theorem i.—In a two-fold system of correlations satisfying any six conditions, the number of correlations having, likewise, two pairs of conjugate points, exceeds the number of those in which a given line corresponds to a given point, by the number of axial correlations in the system which have an axis passing through the latter point.†

Theorem ii.—In a two-fold system of correlations satisfying any six given conditions, the number of correlations having, likewise, two pairs of conjugate lines exceeds the number of those in which a given point corresponds to a given line, by the number of central correlations in the system which have a centre situated on the latter line.‡

In proof of the first of the above theorems (the second follows therefrom by the Principle of Duality) it may be observed—*first*, that, although the *character* of some of the correlations which satisfy the six conditions, and have likewise the two pairs of conjugate points $\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}$, may possibly change when two of these points, say A_1 and B_1, are allowed to coincide, their *number* will remain unaltered; *secondly*, that the line which corresponds to any point $A_1 \equiv B_1$ always passes through the conjugates A_2, B_2 of that point, no matter whether the correlation satisfying the six given conditions be an ordinary one, a central one, or an axial one, provided, in the last case, the axis do not pass through $A_1 \equiv B_1$;

* It is due to Prof. Schroeter to state that, in his memoir entitled "Problematis geometrici ad superficiem secundi ordinis per data puncta construendam spectantis solutio nova," which was published at Breslau in 1862 as a professional dissertation, and afterwards in "Crelle's Journal" (Vol. 62, p. 215), he was, I believe, the first to prove that there is one, and only one, solution in each of the cases: (3020), (2040), (1060), and (0080).

† The excess in question is obviously equal to the class of one of the two *envelopes of axes* of axial correlations satisfying the six conditions (art. 66).

‡ The excess in question is equal to the order of one of the two *loci of centres* of central correlations satisfying the six conditions (art. 66).

and *lastly*, that in every axial correlation which, besides satisfying the six conditions in question, has an axis σ_1 passing through a given point $A_1 \equiv B_1$, the line corresponding to that point, being an *indeterminate line through a determinate point* of the associated axis σ_2, will not, in general, pass through *two* given points A_2, B_2, although each of these points, as indeed any point whatever of the plane, may properly be regarded as conjugate to $A_1 \equiv B_1$.

In illustration of the first of these theorems, it may be noted that the number of solutions [4000], as also the number [3100], is the same as the number $[3020] = 1$ (art. 42), because the system of correlations satisfying the six conditions (3000) does not contain any axial ones having an axis passing through an arbitrary point (art. 69). Again, the number [2200] is zero, or less by one than the number $[2120] = 1$, because amongst the correlations which satisfy the six conditions (2100) (art. 70) there is, clearly, an axial one whose axis passes through an arbitrary point of the second plane; it passes also through the given point in that plane, and has, for associate, the line joining the two given points in the first plane.

I may observe, in conclusion, that by means of the above two theorems we might, without difficulty, deduce successively from the well known result $[4000] = 1$, or from the still simpler one $[2200] = 0$, all those, contained in the table of art. 42, in which an even number of pairs of conjugate points or conjugate lines are involved.

On the Free Motion of a Solid through an Infinite Mass of Liquid.

By Prof. H. Lamb, M.A. (Adelaide).

[*Read May* 10, 1877.]

Let us suppose that we have a solid body of any form immersed in an infinite mass of perfect liquid, that motion is produced in this system from rest by the action of any set of impulsive forces applied to the solid, and that the system is then left to itself. The equations of motion of a body under these circumstances have been investigated independently by Thomson* and by Kirchhoff,† and completely integrated for certain special forms of the body. The object of this communication is, in the first place, to examine the various kinds of permanent or *steady* motion of which the body is capable, without making any restrictions as to its form or constitution; and, in the second, to

* *Phil. Mag.*, Nov. 1871.

† *Crelle*, Band 71 (1869—70). See also *Vorlesungen über Mechanik*, by the same Author, ch. xix.

show that, when the initiating impulses reduce to a couple only, the complete determination of the motion can be made to depend upon equations identical in form with Euler's well-known equations of motion of a perfectly free rigid body about its centre of inertia, although the interpretation of the solution is naturally more complex. It will be seen that a very liberal use has been made, throughout, of the ideas and the nomenclature of the Theory of Screws, as developed and established by Dr. Ball.

1. Whatever be the motion at any instant, we may suppose it to have been produced instantaneously from rest by a properly chosen system of impulsive forces applied to the body. This system when reduced, after the manner of Poinsot, to a force and a couple whose axis is in the direction of the force (or, in Dr. Ball's nomenclature, to an impulsive *wrench*), is called by Thomson* the "impulse" of the motion at the instant under consideration. The same writer has shewn, in a manner which admits of considerable simplification, that when no external impressed forces act, the impulse is constant in every respect throughout the motion. This is expressed analytically as follows:— Let us take a system of rectangular axes x, y, z fixed in the body, and let u, v, w be the velocities of the origin parallel to, p, q, r, the angular velocities of the body about, the instantaneous positions of these axes. Further, let $\xi, \eta, \zeta, \lambda, \mu, \nu$ be the components, relative to the same axes, of the force- and couple-resultants of the impulse. The following equations† express that these latter quantities only vary in consequence of the motion of the axes to which they are referred:

$$\left.\begin{aligned} \frac{d\xi}{dt} = r\eta - q\zeta, \quad \frac{d\eta}{dt} = p\zeta - r\xi, \quad \frac{d\zeta}{dt} = q\xi - p\eta, \\ \frac{d\lambda}{dt} = w\eta - v\zeta + r\mu - q\nu, \quad \frac{d\mu}{dt} = u\zeta - w\xi + p\nu - r\lambda, \\ \frac{d\nu}{dt} = v\xi - u\eta + q\lambda - p\mu \end{aligned}\right\} \ldots\ldots\ldots (1).$$

The motion of the fluid being entirely due to that of the solid, the total energy, T, of the motion is a quadratic function of u, v, w, p, q, r, with constant coefficients; so that we may write

$$\begin{aligned} 2\mathrm{T} = {} & \mathrm{A}u^2 + \mathrm{B}v^2 + \mathrm{C}w^2 + 2\mathrm{A}'vw + 2\mathrm{B}'wu + 2\mathrm{C}'uv \\ & + \mathrm{P}p^2 + \mathrm{Q}q^2 + \mathrm{R}r^2 + 2\mathrm{P}'qr + 2\mathrm{Q}'rp + 2\mathrm{R}'pq \\ & + 2p\,(\mathrm{L}u + \mathrm{M}v + \mathrm{N}w) \\ & + 2q\,(\mathrm{L}'u + \mathrm{M}'v + \mathrm{N}'w) \\ & + 2r\,(\mathrm{L}''u + \mathrm{M}''v + \mathrm{N}''w) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2), \end{aligned}$$

* *Edin. Trans.*, Vol. xxv., 1869.

† See a paper by Mr. Hayward, *Camb. Trans.*, Vol. x.

where A, B, C, &c. are certain constants depending on the shape of the body and the distribution of matter in its interior, determinable by a transcendental analysis whose nature has been exhibited in a very elegant manner by Kirchhoff (*l. c.*).

The values of ξ, η, ζ, &c. in terms of u, v, w, &c. are then given by the formulæ

$$\xi, \eta, \zeta, \lambda, \mu, \nu = \frac{d\mathrm{T}}{du}, \frac{d\mathrm{T}}{dv}, \frac{d\mathrm{T}}{dw}, \frac{d\mathrm{T}}{dp}, \frac{d\mathrm{T}}{dq}, \frac{d\mathrm{T}}{dr} \quad \ldots\ldots\ldots\ldots (3),$$

as may be shewn by processes identical with those employed in Thomson and Tait's *Natural Philosophy*, Art. 313.

2. Kirchhoff has observed that the equations (1) are satisfied by u, v, w all constant, and p, q, r zero, provided we have

$$\frac{\frac{d\mathrm{T}}{du}}{u} = \frac{\frac{d\mathrm{T}}{dv}}{v} = \frac{\frac{d\mathrm{T}}{dw}}{w},$$

that is, provided the velocity of which u, v, w are the components be parallel to one of the principal axes of the ellipsoid

$$\mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 + 2\mathrm{A}'yz + 2\mathrm{B}'zx + 2\mathrm{C}'xy = \text{const.} \quad \ldots\ldots\ldots (1a).$$

There exist, therefore, for every body three mutually perpendicular directions of permanent translation; that is to say, if the body be set in motion parallel to one of these directions, without rotation, it will continue to move in this manner. It is worthy of remark that these directions depend on the *shape* of the body, and not at all on the distribution of matter in its interior, or on the density of the surrounding fluid.

The above, although the simplest, are not the only permanent or steady motions of which the body is capable. In fact, the conditions that (1) should be satisfied by constant values of u, v, w, &c. are equivalent to only *four* independent relations between the five ratios $u : v : w : p : q : r$; viz., the first three of equations (1) give

$$\frac{\xi}{p} = \frac{\eta}{q} = \frac{\zeta}{r} \; [= h, \text{ say}] \quad \ldots\ldots\ldots\ldots\ldots (4);$$

and the last three give, when combined with these,

$$\frac{\lambda - hu}{p} = \frac{\mu - hv}{q} = \frac{\nu - hw}{r} \; [= k, \text{ say}] \quad \ldots\ldots\ldots\ldots (5).$$

There exists, then, a simply-infinite system of possible steady motions. These motions consist each of a *twist* about a certain *screw*; and the axes of this system of screws lie on a certain skew surface, which is, of course, fixed relatively to the body.

The axis of the impulsive wrench necessary to produce motion ac-

cording to one of these screws is the same with that of the screw. For (4) express that the axes of the screw and wrench are parallel, whilst the conditions that they should be coincident may be easily shewn to be

$$\frac{\lambda p-\xi u}{p^2}=\frac{\mu q-\eta v}{q^2}=\frac{\nu r-\zeta w}{r^2},$$

and we see, from (4) and (5), that each of these fractions is, in fact, equal to k. Indeed, it is otherwise obvious that the motion of the body about any assigned screw is steady when, and only when, it does not affect the configuration of the "impulse," fixed in space, relatively to the body; that is, when the axes of the screw and corresponding impulsive wrench coincide.

The *pitches*, however, of the screw and wrench are in general different. Calling κ the pitch of the screw, κ' that of the wrench, we have

$$\kappa'=\frac{\lambda\xi+\mu\eta+\nu\zeta}{\xi^2+\eta^2+\zeta^2}=\frac{pu+qv+rw}{p^2+q^2+r^2}+\frac{k}{h},$$

by (4) and (5), so that $k=h\,(\kappa'-\kappa)$ .....(6).

The expression for the kinetic energy may now be written

$$\begin{aligned}2\mathrm{T}&=\xi u+\&\mathrm{c.}+\lambda p+\&\mathrm{c.}\\&=(\kappa+\kappa')\,h\omega^2,\end{aligned}$$

where ω is the angular velocity $\sqrt{p^2+q^2+r^2}$.

3. If in (4) and (5) we substitute the values of ξ, η, ζ, &c. from (2) and (3), we get the following system of equations:

$$\left.\begin{array}{lll}(\mathrm{P}-k)\,p+\mathrm{R}'q+\mathrm{Q}'r+(\mathrm{L}-h)\,u+\mathrm{M}v+\mathrm{N}w=0, & \\ \quad\&\mathrm{c.} & \quad\&\mathrm{c.}, \\ (\mathrm{L}-h)\,p+\mathrm{L}'q+\mathrm{L}''r+ & \mathrm{A}\,u+\mathrm{C}'v+\mathrm{B}'w=0, \\ \quad\&\mathrm{c.} & \quad\&\mathrm{c.}\end{array}\right\}\ \ldots\ldots\ldots(7),$$

whence, on elimination of the ratios $u:v:w:$ &c., we derive the following relation between h and k:

$$\left|\begin{array}{cccccc}\mathrm{P}-k, & \mathrm{R}', & \mathrm{Q}', & \mathrm{L}-h, & \mathrm{M}, & \mathrm{N} \\ \mathrm{R}', & \mathrm{Q}-k, & \mathrm{P}', & \mathrm{L}', & \mathrm{M}'-h, & \mathrm{N}' \\ \mathrm{Q}', & \mathrm{P}', & \mathrm{R}-k, & \mathrm{L}'', & \mathrm{M}'', & \mathrm{N}''-h \\ \mathrm{L}-h, & \mathrm{L}', & \mathrm{L}'', & \mathrm{A}, & \mathrm{C}', & \mathrm{B}' \\ \mathrm{M}, & \mathrm{M}'-h, & \mathrm{M}'', & \mathrm{C}', & \mathrm{B}, & \mathrm{A}' \\ \mathrm{N}, & \mathrm{N}', & \mathrm{N}''-h, & \mathrm{B}', & \mathrm{A}', & \mathrm{C}\end{array}\right|=0\ \ldots(8).$$

If we assign any arbitrary value whatever to h, this equation gives us three corresponding values of k; and, taking any one of these and substituting in (7), we get a system of linear equations to determine the values of the ratios $u:v:w:$ &c., and thence the axis and pitch

of the corresponding permanent screw. The three values of k corresponding to any given value of h are all real; and the axes of the three corresponding screws are mutually at right angles, although they do not in general intersect. To prove these statements, let the axes of x, y, z (fixed in the body) be chosen parallel to the three directions of permanent translation, so that we have A′, B′, C′ all zero. The last three of equations (7) then give us at once u, v, w in terms of p, q, r; whence, substituting in the first three, we get to determine $p : q : r$ and k three equations of the form

$$\left.\begin{aligned} \mathfrak{P}p + \mathfrak{R}'q + \mathfrak{Q}'r &= kp \\ \mathfrak{R}'p + \mathfrak{Q}q + \mathfrak{P}'r &= kq \\ \mathfrak{Q}'p + \mathfrak{P}'q + \mathfrak{R}r &= kr \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9);$$

and the truth of the above statements then follows by known theorems. It also appears that two values of k corresponding to the same value of h cannot coincide unless h satisfy *two* independent conditions, which is in general impossible. This shews that the skew-surface which is the locus of the system of permanent screws consists of three distinct sheets.

If, on the other hand, in (8) we assign any arbitrary value to k, we get an equation of the sixth degree to determine h. The roots of this equation, when all real, determine by means of (7) a system of six *co-reciprocal* screws; that is to say, a wrench of the type of any one of the screws does no work on a body moving according to any other screw of the system. Let the suffixes $_1$ and $_2$ relate to two screws of the system; the condition that these should be reciprocal is

$$p_1u_2 + q_1v_2 + r_1w_2 + p_2u_1 + q_2v_1 + r_2w_1 = 0 \quad \ldots\ldots\ldots\ldots (10).$$

Now in the six equations (7) suppose the suffix $_1$ attached to the symbols h, u, v, w, &c.; then, multiplying these equations by p_2, q_2, r_2, u_2, v_2, w_2 in order, and adding, we find

$$h_1 (p_2u_1 + q_2v_1 + r_2w_1 + p_1u_2 + q_1v_2 + r_1w_2) \quad \ldots\ldots\ldots\ldots (11),$$

equal to an expression symmetrical with respect to $_1$ and $_2$, viz. to

$$(\mathrm{P} - k)\, p_1p_2 + \mathrm{P}'(q_1r_2 + q_2r_1) + \&c.,$$

which must therefore also be equal to

$$h_2 (p_2u_1 + q_2v_1 + r_2w_1 + p_1u_2 + q_1v_2 + r_1w_2) \quad \ldots\ldots\ldots\ldots (12).$$

But, h_1, h_2 being in general different, (11) and (12) cannot be identical unless (10) is satisfied.

4. Of the infinite number of groups of six co-reciprocal screws which may be chosen out of the system of permanent screws belonging to the body, there is one deserving of special notice. It is that in which each screw of the group has the same pitch as the impulsive wrench neces-

sary to produce motion about that screw only. The condition for this is, by (6), $k=0$, and there is no difficulty in showing that in this case the roots of (8) are all real. Consider, in fact, the determinant on the left-hand side of (8), and the subsidiary determinants obtained from it by erasing, first, the first row and column; next, the first two rows and columns; and, again, the first three rows and columns; and suppose k put $=0$ in each. We have thus a series of four determinants, each of which is symmetrical; and the argument employed by Salmon, *Higher Algebra*, Art. 44, shews that, when h is varied, no change in the number of variations of sign in the series can take place, except when h passes through a root of (8). Now, when $h=0$, the sign of each member of the series is $+$, by virtue of the well-known conditions necessary to ensure that the expression for 2T in (2) shall be positive whatever values be assigned to the variables u, v, w, &c. Again, when $h=\pm\infty$, the signs are $-+-+$; that is, there are three variations of sign in the series.

Hence the equation (8) (with $k=0$) has six real roots, three positive and three negative. The six determinate screws thus obtained (of which three are right-handed, three left-handed) may be called the six *principal screws of inertia* of the dynamical system. The name is due to Dr. Ball, who has pointed out the existence of such a set of screws for the case of a perfectly free rigid body.

5. Out of the infinite system of permanent screws we may form, besides the sets of co-reciprocal screws considered in Art. 3, sets of another kind, of at least equal importance. Two screws, such that the impulsive wrench necessary to produce motion according to one of them does no work on a body moving according to the other, may be called *conjugate*. We may obtain sets of six mutually conjugate permanent screws in the following manner:—Let us write, in (8), $k=\theta h$; this equation then gives, corresponding to any arbitrary value of θ, six values of h. When these are all real, we have in (7) a system of linear equations to determine the ratios $u : v : w$: &c. in the several screws of the set. Let the suffixes $_1$ and $_2$ refer to any two screws of the system. The condition that these should be conjugate is that the expression

$$\xi_1 u_2+\eta_1 v_2+\zeta_1 w_2+\lambda_1 p_2+\mu_1 q_2+\nu_1 r_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

should vanish. Now, by (4) and (5), we may write this expression in the form

$$h_1 \{ p_1 u_2+q_1 v_2+r_1 w_2+p_2 u_1+q_2 v_1+r_2 w_1 + \theta (p_1 p_2+q_1 q_2+r_1 r_2) \} \quad \ldots\ldots\ldots\ldots\ldots (14).$$

But when for ξ_1, η_1, ζ_1, &c. we substitute their values obtained from (3) and (2), (13) assumes a form which is, of course, symmetrical with respect to $_1$ and $_2$. Hence it must also be equal to an expression iden-

tical with (14), except that h_2 is written for h_1. Since h_1, h_2 are in general different, this requires that (14), and therefore (13), should be zero. The six screws corresponding to any given value of θ form, therefore, a mutually conjugate system.

Among these systems is included, of course, that of the six principal screws of inertia, obtained by making $\theta=0$.

The system which comes next in importance is found by making $\theta=\infty$. The equation (8) has then three of its roots infinite, and three zero. The infinite roots correspond to the three permanent translations of Art. 1; the zero roots give three screws such that, if the body be struck by an impulsive *couple* about the axis of one of these, it will be set in permanent motion about that screw.

6. The existence of this last set of three screws may be verified by observing that the equations (1) are satisfied by $\xi, \eta, \zeta = 0$, provided we have at the same time

$$\frac{\lambda}{p} = \frac{\mu}{q} = \frac{\nu}{r} \quad [= k].$$

The directions of their axes, relatively to the three directions of permanent translation, are determined by the values of $p : q : r$, obtained from (9), where we now have

$$\mathfrak{P} = \mathrm{P} - \frac{\mathrm{L}^2}{\mathrm{A}} - \frac{\mathrm{M}^2}{\mathrm{B}} - \frac{\mathrm{N}^2}{\mathrm{C}},$$

$$\mathfrak{P}' = \mathrm{P}' - \frac{\mathrm{L}'\mathrm{L}'}{\mathrm{A}} - \frac{\mathrm{M}'\mathrm{M}''}{\mathrm{B}} - \frac{\mathrm{N}'\mathrm{N}''}{\mathrm{C}},$$

&c. &c.

We infer that the axes are mutually at right angles, and that the three corresponding values of k, which I denote by k_1, k_2, k_3, are real. By putting $h=0$ in (8), we see that k_1, k_2, k_3 are, moreover, essentially positive, as is otherwise dynamically evident.

The interest of this set of screws rests on the fact that, when they are known, the motion of the body, when set in motion by any impulsive *couple* whatever, can be completely determined. In fact, let the initiating couple be resolved into three components F, G, H about the axes of the three screws, and let κ_1, κ_2, κ_3 be the pitches of the latter. Then the motion of the body at any subsequent instant is made up of motions s_1, s_2, s_3 about these screws; viz., s_1 denotes a rotation with angular velocity s_1 about the axis of the first screw, accompanied by a translation with velocity $\kappa_1 s_1$ parallel to this axis; and similarly for s_2, s_3. The initial values of these quantities are given by the formulæ

$$k_1 s_1 = \mathrm{F}, \quad k_2 s_2 = \mathrm{G}, \quad k_3 s_3 = \mathrm{H};$$

and their values at any subsequent time t by the equations

$$\left.\begin{aligned} k_1 \frac{ds_1}{dt} &= (k_2 - k_3)\, s_2 s_3 \\ k_2 \frac{ds_2}{dt} &= (k_3 - k_1)\, s_3 s_1 \\ k_3 \frac{ds_3}{dt} &= (k_1 - k_2)\, s_1 s_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

These equations may be proved by putting $\xi, \eta, \zeta = 0$, and $p, q, r = s_1, s_2, s_3$, respectively, in (1); they express simply that the couple whose components are $k_1 s_1$, $k_2 s_2$, $k_3 s_3$ is fixed in space. They are identical in form with Euler's equations of rotation of a perfectly free rigid body, which are, indeed, a particular case of (15), obtained by supposing the density of the fluid surrounding the solid to be zero. The solution of (15) may therefore be regarded as known; but its interpretation requires a knowledge of the properties of the *ternary screw-complex*, some of which are enumerated in the next Article.

7. The screws about which motion can be produced by the action of an impulsive couple only form a doubly-infinite system, *coordinate* (to use Dr. Ball's expression) with the three permanent screws above-mentioned. The properties of this system, which are the same as those of the general *ternary screw-complex*, may be obtained as follows:—The three permanent screws may be considered as lying along alternate edges of a rectangular parallelepiped; let us take as axes of coordinates lines drawn through the centre of this parallelepiped parallel to its edges. The equations to the axes of the three fundamental screws are then of the forms

$$\begin{aligned} y &= -g, \quad z = h; \\ z &= -h, \quad x = f; \\ x &= -f, \quad y = g \end{aligned}$$

respectively; and if κ_1, κ_2, κ_3 be the pitches of these screws, a unit motion according to any other screw of the coordinate system (whose direction-cosines are, say, l, m, n) is equivalent to rotations l, m, n about the axes of x, y, z, and to a translation of the origin whose components, parallel to the same axes, are

$$\kappa_1 l - mh - ng, \quad \kappa_2 m - nf - lh, \quad \kappa_3 n - lg - mf$$

respectively. This translation is, therefore, perpendicular to the plane conjugate, with respect to the quadric

$$\kappa_1 x^2 + \kappa_2 y^2 + \kappa_3 z^2 - 2fyz - 2gzx - 2hxy = \text{const.} \quad \ldots\ldots\ldots\ldots (16),$$

to the direction (l, m, n) of the axis of the screw, and the pitch κ of the latter is given by the formula

$$\kappa = \kappa_1 l^2 + \kappa_2 m^2 + \kappa_3 n^2 - 2fmn - 2gnl - 2hlm;$$

i. e., it is inversely proportional to the square of the diameter of (16) drawn in the same direction. The principal axes of (16) coincide, therefore, with the axes of three screws of the system. The properties of the latter are then best investigated by taking the three axes last-mentioned as axes of coordinates. If a, b, c be the pitches of the corresponding screws, the equations to the axis of the coordinate screw whose components are l, m, n, are

$$\frac{al+mz-ny}{l} = \frac{-zl+bm+nx}{m} = \frac{yl-xm+cn}{n} \quad \ldots\ldots\ldots (17),$$

and its pitch is $\kappa = al^2+bm^2+cn^2$.

Through any point (x, y, z) pass the axes of three screws of the system; for we have, to determine $l : m : n$ and κ, the equations

$$\left.\begin{aligned} (a-\kappa)\,l + \quad zm - \quad yn &= 0 \\ -zl + (b-\kappa)\,m + \quad xn &= 0 \\ yl - \quad xm + (c-\kappa)\,n &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (18).$$

Hence κ is given by the cubic

$$(a-\kappa)(b-\kappa)(c-\kappa)+(a-\kappa)\,x^2+(b-\kappa)\,y^2+(c-\kappa)\,z^2 = 0 \quad \ldots (19).$$

The three values of κ hence derived, of which two are imaginary when (x, y, z) lies beyond a certain region in the neighbourhood of the origin, give, on substitution in (18), three determinate sets of values of the ratios $l : m : n$. The form of (18) shews that all the screws of the system having a given pitch lie on a certain hyperboloid. The axes of all the screws of the complex may therefore be obtained by taking one system of generators of each of the series of hyperboloids obtained by giving κ, in (19), all values intermediate between the algebraically greatest and least of the quantities a, b, c. They may also be found in the following manner :—Construct the quadric

$$ax^2+by^2+cz^2 = e^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (20),$$

e being any arbitrary quantity; and from the origin O draw any radius vector OP meeting (20), or the conjugate surface (obtained by changing the sign of e), in P. The pitch κ of that screw of the complex which is parallel to OP is $\kappa = \dfrac{e^3}{\mathrm{OP}^2}$. Now let OM be the perpendicular from O on the tangent plane at P, and along the normal through O to the plane POM measure off a length $\mathrm{ON} = \kappa \tan \mathrm{POM}$. The *sense* in which this line is to be measured is that in which a rotation of the same sign as κ about OP would tend to carry the point M. A line through N, parallel to OP, will be the axis of a screw of the complex, of pitch κ.

It is easily seen that the point whose coordinates are $(c-b)\,mn$, $(a-c)\,nl$, $(b-a)\,lm$ lies on the straight line (17); in fact, that it is

the point N of the above construction, the rest of the proof of which then follows without difficulty.

We see that the axes of the various screws of the complex all pass within a certain finite distance of O, so that at a great distance from O they present the appearance of a system of straight lines radiating from a point.

The properties of the complex are determined by the quadric (20). To find the form and position of this surface, we may proceed as follows:—Starting from the general expression for 2T in Art. 1, we first change the directions of the axes of coordinates so as to coincide with the principal axes of $(1a)$, so that we now have $A', B', C' = 0$. Next, transfer the origin to the point whose coordinates are $\frac{1}{2}\left(\frac{M''}{B}-\frac{N'}{C}\right)$, $\frac{1}{2}\left(\frac{N}{C}-\frac{L''}{A}\right)$, $\frac{1}{2}\left(\frac{L'}{A}-\frac{M}{B}\right)$; the new values of L′, L″, &c. then satisfy the following conditions:—

$$\frac{M''}{B}=\frac{N'}{C},\quad \frac{N}{C}=\frac{L''}{A},\quad \frac{L'}{A}=\frac{M}{B}.$$

The new origin is the point O of the present Article.

The equation to the quadric (20), referred to the new axes, is

$$\frac{L}{A}x^2+\frac{M'}{B}y^2+\frac{N''}{C}z^2+\left(\frac{M''}{B}+\frac{N'}{C}\right)yz+\left(\frac{N}{C}+\frac{L''}{A}\right)zx+\left(\frac{L'}{A}+\frac{M}{B}\right)xy = -e^3 \ldots\ldots (21).$$

The theorems of the present Article are not, I believe, all new;* but I am not aware to what extent the above investigations of them are so.

8. We are now prepared to construct the solution of (15). The motion of the body may be considered as made up of two parts, the motion of the point O above defined, and the motion relative to O. The latter is given at once by Poinsot's well-known construction; viz., if we construct the ellipsoid (fixed in the body), whose equation, referred to axes through O parallel to the three fundamental screws, is

$$k_1x^2+k_2y^2+k_3z^2=\mathfrak{m}e'^4,$$

(where $\mathfrak{m}$, e', are two quantities of the nature of a mass and a line, respectively), and the plane (fixed in space) whose equation relative to the *initial* positions of the above axes is

$$Fx+Gy+Hz=\sqrt{2T\,.\,\mathfrak{m}}\,.\,e'^2,$$

* See the Postscript to Dr. Ball's paper in the *Phil. Trans.* for 1874; also Art. 31 of a paper by Prof. Everett, "On a New Method in Statics and Kinematics," in the *Messenger of Mathematics*, 1874. I may perhaps be allowed to mention that my knowledge of the Theory of Screws is derived almost entirely from the two sources here indicated; I have not seen, and have not access to, Dr. Ball's papers in the *Trans. R. I. Academy*.

the *angular* motion of the body is obtained by making the ellipsoid roll on the plane with angular velocity $\sqrt{\frac{2T}{m}} \cdot \frac{OI}{e'^2}$ about the radius vector drawn to the point of contact I.

The representation of the actual motion may then be completed by impressing on the whole system of rotating ellipsoid and plane a motion of translation equal to that of O, obtained as follows. Construct the quadric (20) or (21), and let OI meet it in P. The motion of O at any instant, is in the direction of the normal OM to the tangent plane to this quadric at P, and its velocity is $\sqrt{\frac{2T}{m}} \cdot \frac{e^3 \cdot OI}{e'^2 \cdot OP \cdot OM}$.

Suppose the body to be set in motion according to one of the three fundamental screws of this and the preceding Articles, and then slightly disturbed, say by an infinitely small impulse. It is plain, from the analogy with the theory of the rotation of a perfectly free solid about its centre of inertia, that if the disturbing impulse reduce to a couple only, the motion is stable about either of the two screws for which k is respectively greatest and least, and unstable about the remaining screw. It is not difficult, with the help of equations (1), to extend these conclusions to the case when the disturbing impulse is any whatever.

9. For particular varieties of the moving solid, the nature and relative configuration of the various screws considered in the preceding theory become, of course, greatly simplified. For instance, suppose the body to have a plane of symmetry, as regards both its form and the distribution of matter in its interior, and let us take this as the plane of xy. It is easy to see that the energy of the motion must be unaltered if we change the signs of w, p, q. This requires that A′, B′, P′, Q′, L, M, L′, M′, N″ should all vanish. The axis of z is then one of the axes of permanent translation; the other two are at right angles in the plane xy. The pitch of each of the three fundamental screws of Art. 6 is zero; the axis of one of them is parallel to z, and those of the other two are in the plane xy, but do not in general intersect the first axis. The quadric (21) reduces to a hyperbolic cylinder.

Again, let us suppose that the body has a second plane of symmetry, at right angles to the former one; and let us take this as the plane of zx. We find, in the same manner as before, that the coefficients C′, R′, N, L″ must now also vanish. The three directions of permanent translation are then parallel to the axes of coordinates. The axis of x is the axis of one of the fundamental screws, (now pure rotations,) and the axes of the other two are perpendicular to this, one in each plane of symmetry, but do not in general intersect it in the same point. The quadric (21) is a hyperbolic cylinder asymptotic to the two planes of symmetry.

Finally, if the body has three planes of symmetry, mutually at right angles, the expression for 2T reduces to the form

$$2T = Au^2 + Bv^2 + Cw^2 + Pp^2 + Qq^2 + Rr^2.$$

In this case the axes of coordinates are the directions of the permanent translations, and also the axes of the three permanent rotations which could be produced by the action of impulsive couples only.

Let us next consider another class of cases, where the body has no plane of symmetry, but has a sort of skew symmetry with respect to a certain axis; viz., it is identical with itself turned through two right angles about that axis. An instance is the screw-propeller of a ship. Let us take the axis above mentioned as axis of x; it is plain that the energy of the motion must be unchanged if we reverse the signs of v, w, q, and r. Hence the coefficients B′, C′, Q′, R′, M, N, L′, L″ must all vanish. The axis of x is then one of the axes of permanent translation; it is also the axis of one of the fundamental screws of Art. 6, the pitch being $-\frac{L}{A}$.

If the body be identical with itself turned through *one* right angle about the same axis (as, *e.g.*, in the case of a four-bladed screw-propeller), the expression (2) must be unchanged when we write v for u and $-u$ for v. This gives $B = C$, $Q = R$, $M' = N''$, $M'' = -N'$; and we see further, that by changing the origin to the point O, defined in Art. 7, we make M″, N′ both vanish; so that (2) now assumes the form

$$2T = Au^2 + B(v^2 + w^2) + Pp^2 + Q(q^2 + r^2) + 2Lpu + 2M'(vq + wr).$$

Any direction in the plane yz is a direction of permanent translation; and any line through O in that plane is the axis of a permanent screw motion (due to an impulsive couple only) of pitch $-\frac{M'}{B}$.

If the body be, moreover, identical with itself turned through a right angle about any axis in the plane yz, we see that (2) must become

$$2T = A(u^2 + v^2 + w^2) + P(p^2 + q^2 + r^2) + 2L(pu + qv + rw).$$

Any direction whatever is then a direction of permanent translation; and any screw whose axis passes through O is a permanent screw. The equations (1) are in this case at once integrable, as has been pointed out by Thomson,* who has also shewn how to construct a body possessing the properties defined by (22).

Postscript.—I give here the equations of motion of a solid, under the circumstances considered in this paper, referred to the six principal

* *Phil. Mag.*, Nov. 1871.

screws of inertia whose existence and properties have been proved in Art. 4. I have not made any application of these equations, but their *form* seems of itself worthy of notice.

Let the coordinates of the six principal screws, referred to axes fixed in the body, be $(a_1, b_1, c_1, l_1, m_1, n_1)$ ..... $(a_6, b_6, c_6, l_6, m_6, n_6)$; viz., l_1, m_1, n_1 are the direction-cosines of the axis of the first screw, a_1, b_1, c_1 the velocities of the origin, parallel to x, y, z, produced by unit of twist-velocity about this axis, and similarly for the other screws of the system. These quantities are connected by 15 relations of the form

$$a_1 l_2 + b_1 m_2 + c_1 n_2 + a_2 l_1 + b_2 m_1 + c_2 n_1 = 0 \quad \ldots\ldots\ldots\ldots \quad (23).$$

The pitches κ_1, κ_2, ... κ_6 of the principal screws are given by the formulæ

$$\kappa_1 = a_1 l_1 + b_1 m_1 + c_1 n_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (24),$$

$$\&c. \qquad \&c.$$

Let us suppose now that the motion of the body at any instant is made up of twist-velocities s_1, s_2, ... s_6 about these screws, and let σ_1, σ_2, ... σ_6 be the force-components of the corresponding impulsive wrenches. We have then, with the notation of Art. 1,

$$\left.\begin{aligned} u &= a_1 s_1 + a_2 s_2 + \ldots + a_6 s_6, \\ v &= b_1 s_1 + \ldots, \\ w &= c_1 s_1 + \ldots, \\ p &= l_1 s_1 + l_2 s_2 + \ldots + l_6 s_6, \\ q &= m_1 s_1 + \ldots, \\ r &= n_1 s_1 + \ldots, \end{aligned}\right\} \ldots (25), \quad \text{and} \quad \left.\begin{aligned} \xi &= l_1 \sigma_1 + l_2 \sigma_2 + \ldots + l_6 \sigma_6, \\ \eta &= m_1 \sigma_1 + \ldots, \\ \zeta &= n_1 \sigma_1 + \ldots, \\ \lambda &= a_1 \sigma_1 + a_2 \sigma_2 + \ldots + a_6 \sigma_6, \\ \mu &= b_1 \sigma_1 + \ldots, \\ \nu &= c_1 \sigma_1 + \ldots, \end{aligned}\right\} \ldots (26).$$

Hence

$$\begin{aligned} 2T &= \xi u + \&c. + \lambda p + \&c. \\ &= 2\kappa_1 \sigma_1 s_1 + 2\kappa_2 \sigma_2 s_2 + \ldots + 2\kappa_6 \sigma_6 s_6, \end{aligned}$$

by virtue of (25), (26), and of the 15 relations of the type (23). Again, multiplying (25) by l_1, m_1, n_1, a_1, b_1, c_1 in order, and adding, we find

$$2\kappa_1 s_1 = l_1 u + m_1 v + n_1 w + a_1 p + b_1 q + c_1 r,$$

and similarly from (26) we derive

$$2\kappa_1 \sigma_1 = a_1 \xi + b_1 \eta + c_1 \zeta + l_1 \lambda + m_1 \mu + n_1 \nu.$$

To form the equations of motion in terms of s_1, s_2, &c., we must substitute for $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$, &c. from (1), in the equation

$$2\kappa_1 \dot{\sigma}_1 = a_1 \dot{\xi} + \&c. + l_1 \dot{\lambda} + \&c.,$$

and, in the result, for ξ, η, ζ, &c. from (26). The final result is rather long, but it is easy to verify that the coefficients of $\sigma_1 s_1$, $\sigma_2 s_2$, $\sigma_3 s_3$, &c. are all zero, as are also those of $\sigma_1 s_2$, $\sigma_2 s_1$, &c. The coefficient of $\sigma_2 s_3$ is found to be

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 & c_2 \\ l_3 & m_3 & n_3 \\ l_1 & m_1 & n_1 \end{vmatrix} + \begin{vmatrix} a_3 & b_3 & c_3 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

which denote by [123]. We have, obviously, $[123] = [231] = [312] = -[132] = -[213] = -[321]$, so that the number of these coefficients is, irrespective of sign, 20.* The equations of motion then become

$$2\kappa_1 \dot{\sigma}_1 = [123]\,\sigma_2 s_3 + [124]\,\sigma_2 s_4 + \&c.$$
$$+ [132]\,\sigma_3 s_2 + [142]\,\sigma_4 s_2 + \&c. \qquad (20 \text{ terms})$$

with five similar equations. If $h_1, h_2, \ldots h_6$ be the six roots of (8), (with $k=0$,) we have $\sigma_1 = h_1 s_1$, $\sigma_2 = h_2 s_2$ &c., so that the above equations may also be written

$$2\kappa_1 h_1 \dot{s}_1 = [123]\,(h_2 - h_3)\, s_2 s_3 + [124]\,(h_2 - h_4)\, s_2 s_4 + \&c.$$
$$\&c. \qquad \&c. \qquad (10 \text{ terms})$$

We verify at once that these have the particular integrals

$$\kappa_1 h_1 s_1^2 + \kappa_2 h_2 s_2^2 + \ldots + \kappa_6 h_6 s_6^2 = \text{const.},$$
$$\kappa_1 h_1^2 s_1^2 + \kappa_2 h_2^2 s_2^2 + \ldots + \kappa_6 h_6^2 s_6^2 = \text{const.}$$

On some Cases of Parallel Motion. *By* Mr. Harry Hart, M.A., Mathematical Instructor at the Royal Military Academy, Woolwich.

[*Read April 12th*, 1877.]

1. Let ABCD (Fig. 1)† be a contra-parallelogram, where $AB = CD = 2ae$, $AC = BD = 2a$, and O, O′ the middle points of AB, CD. Then P, the intersection of AC, BD, describes an ellipse relatively to AB or DC, the major axis being $2a$, the eccentricity e. Let $2b =$ minor axis, so that $b^2 = a^2(1-e^2)$.

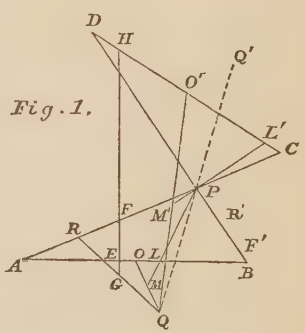

Fig. 1.

On the normal QPQ′ let two points Q, Q′ be taken such that PQ = PQ′ = semi-diameter conjugate to P. It can easily be shown that $OQ = O'Q' = a-b$, $OQ' = O'Q = a+b$, and that AB, DC bisect the angles QOQ′, QO′Q′ respectively.

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* All theoretically expressible, however, in terms of the six pitches $\kappa_1, \kappa_2, \ldots \kappa_6$ and of three other independent quantities.

† To render the figure as simple as possible, some of the lines described in the text have been omitted.

Bisect OQ, O′Q in M, M′; then MPM′Q forms a contra-parallelogram, MP meeting AB in a point L such that $MO = MQ = ML = \frac{1}{2}(a-b)$, and M′P′ produced meeting CD in L′, where $M'O' = M'Q = M'L' = \frac{1}{2}(a+b)$, OLQ, O′L′Q being right angles.

Hence, if the contra-parallelogram formed by four bars OQ, O′Q, PLM, M′PL′ be jointed at M, M′ to that formed by AB, DC, AC, BD, we shall have in the free motion of the linkage P always moving along AC and BD, L along AB, L′ along CD.

2. Draw QF, QF′ perpendicular to AP, BP; then $PF = PF' = b$. Also make $AR = BR' = a-b = OQ$. Then

$$\begin{aligned} RF^2 = (AP-a)^2 &= a^2 - AP\,(2a-AP) \\ &= a^2 - AP\,.\,BP \\ &= a^2 - PQ^2 \\ &= a^2 - b^2 - QF^2 \\ &= a^2e^2 - QF^2, \end{aligned}$$

tha is, $$RQ = ae = AO.$$

Similarly, $QR' = ae$. Thus the four lines QO, QO′, QR, QR′ form with ABCD five contra-parallelograms, viz., RAOQ, R′BOQ, QRCO′, QR′DO′, ABCD.

3. Bisect RQ in G, and draw FG meeting AB in E. Then

$$PL = PM - ML = b = PF,$$

and $$\widehat{PLO} = 180° - \widehat{OLM} = 180° - \widehat{LOM} = \widehat{AOQ} = \widehat{ARQ} = \widehat{PFE}.$$

Thus $EL = EF$, and PE bisects the angle APL;

therefore $$AE : EL :: AP : PL;$$

componendo, $$AE + EL : EL :: AP + PL : PL,$$

or $$ae + OL : a + RF + b :: EL : b;$$

but $$OL : RF :: a-b : ae$$
$$:: ae : a+b,$$

hence $$EL = \frac{abe}{a+b} = EF.$$

If then the contra-parallelogram formed of four bars AR, RQ, QO, OA, be jointed at G, M to those forming the kite-shaped quadrilateral EFPL, the links AR, AO will always be collinear with the links FP, EL. Further, by stripping off the bars AR, AO, we have left a linkage of six bars; PF being fixed, RQ moves with its extremities R, Q on two lines at right angles, that is, has a tram motion. OQ has a similar motion on fixing EL.

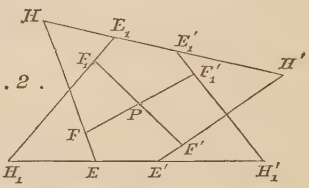
4. Bisecting QR′ in G′, G′F′ meets AB in E′, where $F'E' = \frac{abe}{a+b} = LE$.

Constructing then the eight-bar linkage QR, QR′, GEF, G′E′F′, PF, PF′, PI, EE′, we see that Q describes a straight line whether we fix PF, PF′, or EE′; R, R′ also moving along PF, PF′.

If EF, E′F′ be produced to meet CD in H, H′, it may be shown in a similar manner that $HF = HL' = L'H' = H'F' = \frac{abe}{a-b}$; and that, if we form the six-bar linkage HH′, EE′, HEG, H′E′G′, GQ, G′Q, on fixing HH′ or EE′, Q describes a straight line.

5. Lastly, a second quadrilateral $H_1H_1'E_1'E_1$ may be formed equal in all respects to HH′E′E; H_1H_1' lying on AB, E_1E_1' on CD; one quadrilateral being the image of the other with respect to the common tangent to the ellipses at P; and since they have the lines FPF_1', $F'PF_1$ in common, the two linkages HH′E′E, $H_1H_1'E_1'E_1$, being connected by the equal links FPF_1', $F'PF_1$, move so that HH′ is collinear with E_1E_1', and H_1H_1' with EE′, as in Fig. 2.

Fig. 2.



P.S.—In the foregoing paper, two cases of parallel motion obtained by means of six-bar linkages, or rather (considering one bar fixed) five-bar linkworks, have been given. I have since found that they are both included in the following general method:—

Let PE, PF be two bars jointed at P to each other, and at E, F to two bars BE, CF, B and C being fixed, the lengths of the four bars satisfying the following condition:

$$PE \cdot EB = PF \cdot FC.$$

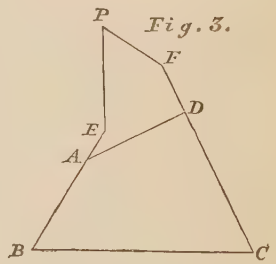
Suppose the bars placed so that the angles at E and F are equal, as in Fig. 3. Take points A, D, in EB, FC, such that

$$PE : EA :: CF : FP,$$

and

$$PF : FD :: BE : EP;$$

Fig. 3.



then it is easily seen that PAD and PBC are two similar triangles, and

$$AD : BC :: PA : PC$$
$$:: PE : FC,$$

that is,

$$AD = \text{constant};$$

so that, if the bars EB, FC be connected by a fifth bar $AD = \frac{BC \cdot PE}{FC}$, the angles at E and F will be equal, and consequently

$$PB^2 - PC^2 = \text{constant},$$

or P describes a straight line perpendicular to BC.

It might also be shown that $PD^2 - PA^2$ is constant, so that, if A, D were fixed points and BC be joined by a bar, then P would describe a straight line perpendicular to AD.

The above may also be expressed thus:—If ABCD be any quadrilateral linkage, $AB = a$, $BC = b$, $CD = c$, $DA = d$, and points E, F be taken in two opposite sides such that

$$AE = a \cdot \frac{d^2}{b^2 - d^2}, \quad FD = c \cdot \frac{d^2}{b^2 - d^2},$$

and $PE \left(= c \cdot \frac{bd}{b^2 - d^2}\right)$, $PF \left(= a \cdot \frac{bd}{b^2 - d^2}\right)$ be connected to them at E, F; then, in the free motion of the linkage, P moves at right angles to AD and BC.

If PE, PF be interchanged, the locus of P is a circle.

June 14*th*, 1877.

Lord RAYLEIGH, F.R.S., President, in the Chair.

The following communications were made to the Society:—

"On Mean Values:" Prof. Crofton.

"On the Canonical Form and Dissection of a Riemann's Surface:" Prof. Clifford.

"On Eisenstein's Theorem:" Prof. H. J. S. Smith.

"Proof that every Algebraic Equation has a Root:" Mr. J. C. Malet (communicated by Dr. Casey).

The following presents were made:—

"Sull' equazione dell' Icosaedro nella risoluzione delle equazioni del quinto grado, nota del S. C. Prof. F. Klein (estratta da una lettera diretta al M. E. Prof. Brioschi), dal Rendiconti del R. Istituto Lombardo," Serie ii., Vol. x., Fasc. ix. e x.

"Theory of Sound," Vol. i., by Lord Rayleigh, F.R.S.: from the Author.

"Bulletin des Sciences," 2^{me} Série, Tome i., Fevr., Mars, 1877.

"Monatsbericht," Jan., Feb., 1877.

"Educational Times," June, 1877.

"Atti della R. Accademia dei Lincei anno cclxxiv.," 1876–77, Serie terza; "Transunti," Vol. i., Fasc. 5°, Ap., 1877; Fasc. 6°, Mai, 1877.

"Crelle," 83 Band, 1es, 2es Heft.

"Proceedings of the Royal Society," Vol. xxvi., No. 179.

"A Continuation of De Damoiseau's Tables of the Satellites of Jupiter to the year 1900," by D. P. Todd, B.A.; Washington, 1876.

"Jahrbuch über die Fortschritte der Mathematik," siebenter Band, Jahrgang 1875 (Heft 2); Berlin, 1877.

"Annali di Matematica," Serie iia, Tomo viiio, Fasc. 2° (Apr. 1877).

"Bulletin de la Société Mathématique de France," Tome v., Nos. 3 and 4; Paris, 1877.

"Astronomical and Meteorological Observations made during the year 1874 at the United States Naval Observatory:" Rear-Admiral C. H. Davis, U.S.N.; Washington, 1877.

"Bücher-verzeichniss von Mayer und Müller," Katalog xxiii.

"Sur le paraboloide des huit droites," and "Sur les surfaces dont les rayons de courbure principaux sont fonctions l'un de l'autre," par M. A. Mannheim: from the Author (Avril 2 & 30, 1877).

"Résumé du Bulletin de l'Académie Royale Danoise des Sciences et des Lettres pour l'année 1877."

"φvelser i Grafisk Statik," af H. G. Zeuthen.

"Særtryk efter Tidsskrift for Mathematik," 4de Række, 1ste Aargang; Kjφbenhavn, 1877.

The following presents were received in the Recess:—

"Atti della R. Accademia dei Lincei," Anno cclxxii., Serie iia, Vols. i., ii., iii., 1873–74, 1874–75, 1875–76; Anno cclxxiii., Serie iia, Vol. iiia, Parte prima e terza, Roma, 1876; Anno cclxxiv., 1876–77, "Transunti," Vol. i., Fasc. 1, Dicembre, 1876; Fasc. 2, Gennajo, 1877; Fasc. 7°, Giugno, 1877; Roma, 1877.

"Constitutiones Lynceorum," An. 1875; Roma, 1876.

"Crelle," 83 Band, 3es, 4es Heft; 84 Band, 1es Heft; Berlin, 1877.

"Monatsbericht," März, April, Mai, Juni, Juli, 1877.

"Proceedings of the Royal Society," Vol. xxvi., Nos. 180—183.

"Bulletin des Sciences Mathématiques et Astronomiques," deuxième Série, Tome i., Avril, Mai, Juin, Juillet, Août, 1877.

"Tables générales des Matières et Noms d'Auteurs," contenus dans la première Série, Tomes i. à xi., 1870–1876.

Catalogue of the Library of the University of London, including the Libraries of George Grote and A. de Morgan; London, 1876.

"Mémoires de la Société des Sciences de Bordeaux," 2e Serie, Tome ii., 1er cahier; Paris, 1877.

List of Members of the Institution of Civil Engineers, June 12, 1877.

"Educational Times," July to November, 1877.

"Mathematical Reprint from Educational Times," Vol. xxvii.

"Journal of Institute of Actuaries," Vol. xx., Part ii., Jan.; Part iii., April; Part iv., July, 1877.

"Lectures on the Elements of Applied Mechanics, comprising—1. Stability of Structures; 2. Strength of Materials;" by M. W. Crofton, F.R.S.: from the Author.

"Bulletin de la Société Mathématique de France," Tome v., Nos. 5, 6, et dernier; Paris, 1877.

"Jahrbuch über die Fortschritte der Mathematik," siebenter Band, Jahrgang 1875, Heft 3; Berlin, 1877.

"Ueber correlative oder reciproke Bündel von Rudolf Sturm," from "Mathematische Annalen," Vol. xii., pp. 254—368.

"Weitere Untersuchungen über das Ikosaeder iii. von F. Klein" (aus den Sitzungsberichten der Phys. M. S. zu Erlangen, Juli 9, 1877).

"Ueber geometrische Erweiterungen des Bezoutschen Fundamentalsatzes von Dr. H. Schubert in Hamburg" (from "Königliche Gesellschaft der Wissenschaften," Juli 7, 1877, No. 17).

"Ueber lineare differential Gleichungen von F. Klein" ("Mathematische Annalen," Vol. xii., pp. 167—179, April, 1877).

"Sur la production du mouvement rectiligne exact, au moyen de tiges articulées," par A. B. Kempe (translated by M. V. Liguine in "Nouvelle Correspondance Mathématique," Tome iii., Mai et Juin, 1877).

"How to Draw a Straight Line," by A. B. Kempe, B.A. ("Nature" Series): from the Author.

"Sopra i sistemi tripli di superficie isoterme e ortogonali," Memoria di Enrico Betti ("Annali di Matematica," Serie ii^a, Tomo viii., pp. 138—145): from the Author.

"Sopra il moto di un sistema di un numero qualunque di punti," Enrico Betti (extract from "Atti della R. Accademia dei Lincei, Transunti," Vol. i., pp. 129, 130).

"Journal of Education," July to November, 1877.

"Boletin de la institucion libre de Enseñanza" (Año 1, Num. 1°—5°, March 7, April 4, May 4, 27, June 17, 1877).

"Catalogue of Scientific Papers" (1864–73), compiled and published by the Royal Society of London, Vol. vii., 1877.

"The Calculus of Chemical Operations; being a method for the investigation, by means of symbols, of the laws of the distribution of weight in chemical change." Part i., "On the Construction of Chemical Symbols," 1866 (Phil. Trans., Part ii., 1866). Part ii., "On the Analysis of Chemical Events," 1877 (Phil. Trans., Part i., 1877). By Sir B. C. Brodie, Bart., F.R.S. London.

"Physical Society of London—Proceedings," Vol. ii., Part ii., March to June, 1877; London, 1877.

"A General Catalogue of Books offered to the Public by B. Quaritch," the Supplement, 1875–1877; London, 1877.

"Weitere Untersuchungen über das Ikosaeder," von F. Klein, in München ("Math. Annalen," Band xii.).

On the Canonical Form and Dissection of a Riemann's Surface.

By Prof. Clifford, F.R.S.

[*Read June 14th*, 1877.]

The object of this Note is to assist students of the theory of complex functions, by proving the chief propositions about Riemann's surfaces in a concise and elementary manner. To this end I assume only certain results of Puiseux, which are put together at the outset.

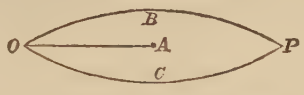
I.

Puiseux's Theory of an n-valued Function.

If two variables s and z are connected by an equation of the form $f(s, z), = (s, 1)^n (z, 1)^m, = 0$, each is said to be an algebraic function of the other. Regarding z as a complex quantity $x+iy$, we represent its value by the point whose coordinates are x, y, on a certain plane. To every point in this plane belongs one value of z, and consequently, in general, n values of s, which are the roots of the equation $f = 0$. The points of the plane may be divided into those at which the n values of s are distinct, and those at which two or more of them are equal. The latter points are finite in number, and correspond to the roots of the equation which is got by equating to zero the discriminant of f in regard to s. If the roots of this equation are distinct, there are $2(n-1)m$ such points, because the discriminant of the equation of the n^{th} order in s is of degree $2(n-1)$ in the coefficients, and these coefficients are of the order m in z. But a point at which r values of s become equal corresponds to an $(r-1)$-fold root of the discriminant-equation.

Let us now consider an arbitrary point O of the plane, corresponding to a value z_0 of z, which is not a root of the discriminant-equation. Then the equation $f(s, z_0) = 0$ will give n different values for s, which we may call $s_1, s_2 \ldots s_n$. If we move along any path from the point O to another point P of the plane, the value of z will change continuously, and each of the quantities $s_1, s_2 \ldots s_n$ will also change continuously. If therefore the path OP does not go through a point where two values become equal, these n quantities will be distinct all the way, and each of the n values of s at P will belong to a definite one of the values of s at O. But if the path goes through such a point, two or more of the n quantities will become equal and then diverge again, so that it will be impossible after that to distinguish them so as to say which of these belongs to a particular one of the values at the point O. We cannot always avoid this difficulty by going *round* the point, for it is found that the values at P to which the values at O correspond may depend upon the path OP, so that the correspondence is different for a path which goes to the right of

the point and for a path which goes to the left of it. When this is the case, the point is called a branch-point. Suppose that, when



we go from O to A, the two values p and q of s at O approach one another and become equal at A; then it is found that the value at P which represents p when we go along the path OBP may represent q when we go along OCP, and *vice versâ*. So that, if we travel along OBPCO, round the point A and back to O, the values p and q will change continuously into one another. If more than two values are equal at A, the corresponding values at O may be cyclically interchanged by a path going round A. We shall assume, however, that only two values become equal at each branch-point; and, moreover, that no branch-point is at an infinite distance.*

A path going along any line from O to very near A, then round A in a very small circle, and then back to O along the same line, will be called a *loop*.

If we start from O and go round any closed curve not including any branch-points, the n values of s at O will be restored in the same order. For the path may be gradually shrunk into a point without crossing any branch-points, so that no two of the n values can become confused at any point of it. The same thing is true if the closed path includes *all* the branch-points. Suppose it a large circle through O; then it may be gradually increased till it coincide with the tangent at O, then curved over on the other side, and shrunk up into a point; and during the whole process the n values will be distinct at every point of the path.

We shall now go on to show that this n-valued function, which we have spread out upon a single plane, may be represented as a *one*-valued function on a surface consisting of n infinite plane sheets, supposed to lie indefinitely near together, and to cross into one another along certain lines. This surface is called a RIEMANN'S surface; we shall demonstrate its existence at the same time that we show how to construct it in the most convenient form.

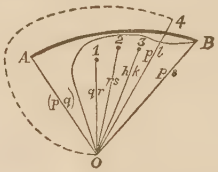
II.

Construction of the Riemann's Surface.—Lüroth's Theorem.

Draw loops from O to all the branch-points, and let the first, A, interchange the values p and q. If we go round all the loops successively, starting with the value p at O, we must, as we have seen, come

* Roots of the discriminant-equation which are not branch-points correspond to double points on the curve $f(s, z) = 0$. Such points behave, in regard to the function s, like two coincident branch-points belonging to the same pair of values, and they have no influence on the connection of the different values of s.

back to that value; but this may happen before we have used all the loops. Let B be the first branch-point after going round which the value p is restored. Draw a line from A to B cutting all the loops which alter p, but none of the others. Then, if we go round any of the branch-points between A and B without crossing the line AB or going round any other branch-points, we shall not alter the value p.



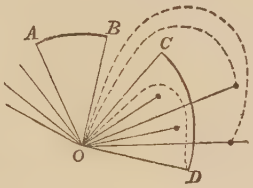
Suppose that A interchanges pq, B interchanges ps, and that the branch-points between A and B are 1, 2, 3, 4, interchanging respectively qr, rs, hk, pl. The value q must in fact be changed into the value p through a longer or shorter series of values; the loops interchanging hk and pl are put in as examples. Now if we go round 4 by the dotted loop passing round outside A, the effect is the same as going in succession round A, 1, 2, 3, 4, 3, 2, 1, A. By the time we have gone round A, 1, 2, 3, we cannot have the value p, for that is first restored by B; and we cannot have the value l, for then 4 would restore the value p. Hence we have some value which is not altered by the loop to 4; and consequently, when we retrace our path, we shall come back to the value p.

Next, let us draw a loop to B which passes within the line AB, but goes round all the included branch-points, as in the figure. The effect of this loop will be to change q into p; for it is the same thing as going round 1, 2, 3, B, 3, 2, 1. Now the effect of 1, 2, 3, B is to change q into p, and this p is not altered in coming back because all the branch-points which alter p are outside the line AB.

Suppose then that all the branch-points of this group which alter p are connected with O by loops going round A, so that they no longer alter p; and that B is connected with O by the loop just described, so that no branch-points are contained in the triangle AOB.

Starting now from this new loop OB, with the value p, let us go round all the loops as before from left to right. We know that when all the loops have been gone round, ending with OA, the value p must be restored. If it is not restored before we have gone round OA, we must draw a line BA cutting all the loops which change the value p but none of the others. But if the value p is restored before we have gone round OA, say after going round OC; then we must draw a new loop to C, going round all the branch-points between A and C except those which change the value p. This new loop will, by our previous

reasoning, change p into q. Hence, if the value p is restored before we have gone round OA, we can make a new loop OC which changes p into q; and this comes next to OB. To those branch-points whose loops have been cut by this new loop we must draw new loops going round to the right of C, so as not to cut OC. The figure comes then into this form:



containing

1) Loops to the left of OA which do not change the value of p, like the dotted loop O4 in the previous figure;
2) Three consecutive loops OA, OB, OC which change p into q;
3) Loops to the right of OC which may or may not change p.

If now we start with the loop OC and proceed to the right, the value p must be restored *before* we have gone round OA; for, starting with OA and going all round, we must restore the value p in the end. Let p then be restored by OD; and draw a line CD cutting all those loops which change p, but none of the others. Replace the loops which change p by new ones going round between B and C; and replace OD by a new loop going outside all the branch-points whose loops do not alter p. The figure now consists of these elements:

1) Two triangles AOB, COD, containing no branch-points, and such that the loops OA, OB, OC, OD interchange p and q;
2) Loops between OB and OC which do not change p;
3) Unknown loops between OD and OA.

About these unknown loops we may make three suppositions.

First, suppose that none of them change p. Then the value p cannot be altered by any closed curve starting from O and returning to it which does not cut either of the lines AB, CD.

Secondly, suppose that some of these loops change p, but that, when we start with the loop OD and go round to the right, the value p is first restored by OA or OC. (It is clear that it cannot be first restored by OB, because the two loops OA, OB, taken together, make no change in any value; nor by any loop between OB and OC, for none of them change p.) Then we must join D with A by a line cutting all the loops which change p, but no others; and B with C by a line cutting none of the loops between OB and OC. In that case the value p cannot be altered by any closed curve starting from O and returning to it which does not cut either of the lines BC, DA.

Thirdly, suppose that the value p is restored *before* we come to OA,

say at OE. Then we must proceed as before, finding a new line EF which shall have the properties of AB or CD. The figure will then consist of three triangles AOB, COD, EOF, containing no branch-points, and such that the loops OA, OB, OC, OD, OE, OF interchange p and q; loops between OB and OC, and between OD and OE, which do not change p; and unknown loops between OF and OA.

It is clear that this process must ultimately stop, and then we shall be left with a finite number of lines such that, if we start from O, follow any continuous path, and come back again, without crossing any of these lines, we shall not alter the value p. The lines are either AB, CD, EF, &c., or else they are BC, DE, &c.; in either case the loops OA, OB, ... interchange p and q.

It follows that, if we take an infinite plane sheet and cut it through along these lines, we may consider a single value of the function s to be attached to every point of the sheet in such a way that this value varies continuously when we move about continuously in the sheet; but there will be different values on the two sides of any cut—namely, we must attach to every point P of the sheet that value of s which changes continuously into p when we go from P to O without crossing any of the cuts. There is only one such value; for if two different paths from O to P gave different values at P, it would be possible to change the value p by means of a closed curve returning to O; and this we have proved not to be the case.

When the lines cut through are AB, CD, ..., the triangles AOB, COD, ... contain no branch points; but when the lines are BC, DE, ..., the triangles BOC, DOE do in general contain branch-points. We may, however, draw new loops to C, E, ... so as to exclude these branch-points, and the new loops will still change p into q. For no closed curve going round B and C so as not to cut BC can change the value p, by what we have already proved; but the loop OB changes p into q, therefore OC must change q into p.

We shall assume then that the cuts are AB, CD, ..., and that the triangles AOB, COD, ... contain no branch-points.

Now let us deal with the value q at O in the same way as we have dealt with the value p. It is first to be observed that a path going round one or more of the lines AB makes no change in *any* value at O; so that, if we agree never to cross these lines, we may leave the branch-points A, B, ... entirely out of consideration.

This being so, let us take a loop which changes q into some other value, say r. There must be such a loop, if the function is more than two-valued; for otherwise the values p, q would form a two-valued algebraic function of z, and the expression $f(s, z)$ would have a factor of the second degree in s.

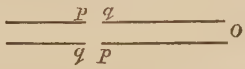
Starting then with this loop, we may proceed in exactly the same

way as before, and draw lines A′B′, C′D′, ... such that a closed curve, starting from O and coming back to it without cutting any of these lines or any of the previously drawn lines, will not alter the value q. Moreover, we shall have drawn loops OA′, OB′, ..., each of which changes q into r, and such that the triangles A′OB′, C′OD′, ... contain no branch-points. And since our previous triangles AOB, COD, ... contained no branch-points, it will not have been necessary to cut through them in drawing the new lines A′B′, C′D′, ...

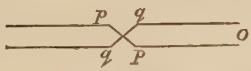
We shall now speak of the first set of lines AB, CD, ... as the lines (pq), and of the second set as the lines (qr).

Let us take two infinite plane sheets, cut them both through along the lines (pq), but only the second one along the lines (qr). To every point of the *first* sheet we will suppose attached that value of s which is arrived at by continuous change of the value p at O; and to every point of the *second*, that value which is arrived at by continuous change of the value q at O.

In each sheet there will be a finite difference in the values on the two sides of each of the cuts (pq); but the value on one side in the upper sheet will be equal to the value on the other side in the lower sheet. At the cut AB, for example, the value continuous with p on the side next to O is equal to the value continuous with q on the side remote from O; because a path taken round A or B from O and back again changes the value p continuously into the value q.



Thus, if we take p, q to denote values at the cut continuous with p, q at O, they will be situated as in the figure, which represents a section across AB perpendicular to the two sheets. If then we make the two sheets cross one another along the lines p, q, as here represented,



then these two values will be continuously distributed on the double-sheeted surface so formed.

We may now continue the process with the value r. We must first find a loop which changes r into some other value, say t, and then proceed as before, taking care not to cross the lines qr. (We may cross the lines pq as often as we please, provided that we have not previously crossed the lines qr; for these lines can have no effect upon r unless it has been previously changed into q.) Thus we shall draw lines rt such that the value r cannot be altered by a closed curve not cutting the lines qr or rt, and having their extremities joined to O by loops

which change r into t. If we take, then, a third sheet, cut it through along the lines qr and rt, and then join it crosswise to our second sheet along the lines qr; the three values pqr may be continuously distributed on this three-sheeted surface.

By proceeding in this way it is clear that we shall construct an n-sheeted surface, the sheets of which are connected chainwise by cross lines, so that the first is connected only with the second, the second with the third, and so on; but there is no direct connection except between consecutive sheets. And the n values of the function may now be attached to the points of this surface, so that one value only belongs to each point, and that in moving this point about on the surface the value belonging to it always changes continuously. Thus, if we start from a given point of the surface (on a given sheet), and travel by any path so as to come back to the same point (on the same sheet), we shall in all cases return to the former value of the function s.

The theorem that the Riemann's surface may be so constructed that the sheets are only connected *chainwise*—*i. e.*, so that there are no cross-lines except between consecutive sheets—is due to Dr. Lüroth.

III.

Clebsch's Theorem.

All the links between successive sheets except the last may be made to consist of one cross-line only.

First, we shall prove that, if there are two or more lines (pq), one of them may be converted into a line (qr).

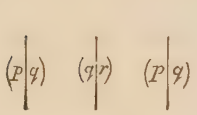


Fig. 1.

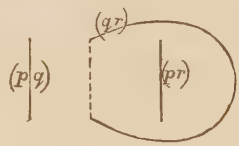


Fig. 2.

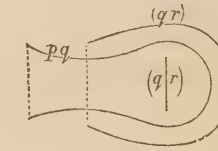


Fig. 3.

The original position of the two lines (pq) and the line (qr) is drawn in Fig. 1. If we move the line qr, keeping, of course, its ends fixed, the effect is to interchange the sheets QR in the area over which it moves; so that, by passing it over the line (pq) on the right, we change this into a line (pr). The position is then as in Fig. 2. If now we pass the remaining line (pq) over this line (pr), we change it into a line (qr); thus we are left with two lines (qr) and one line (pq).

In this way we may convert all but one of the lines (pq) into lines (qr). Then we may convert all but one of the lines (qr) into lines

(rs); and so on. Then the first $n-1$ sheets will be connected chainwise by one cross-line each, and the last two by all the remaining cross-lines.

The Riemann's surface is now said to be in its canonical form.

The process of transformation may be made clearer by looking at a section of the three sheets by a plane perpendicular to them cutting the lines pq, qr, pq.

Fig. 1.

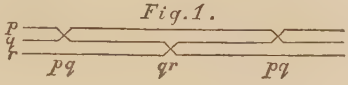


Fig. 2.

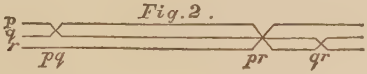
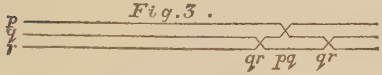


Fig. 3.



IV.

Transformation of the Riemann's Surface.

The Riemann's surface now consists of n infinite plane sheets, such that the sheet 1 is connected with 2 by a single cross-line, 2 with 3 by another cross-line, and so on; but $(n-1)$ with (n) by a number of cross-lines which we shall call $p+1$. Thus the whole number of cross-lines is $n-2+p+1=n+p-1$. If w is the number of branch-points, this is twice the number of cross-lines, or $w=2(n+p-1)$. Hence $p=\frac{1}{2}w-n+1$.

Let now this n-fold plane be inverted in regard to any point outside it, so that it becomes an n-fold sphere passing through the point. Any two successive sheets of the sphere will be connected by one cross-line, except the two outside sheets, which are connected by $p+1$ cross-lines.

To every point of this n-sheeted spherical surface will correspond one value of the function s, namely, that which belongs to the corresponding point upon the n-fold plane. As for the centre of inversion, it is to be regarded as n distinct points upon the several sheets, corresponding to the n values of s when $z=\infty$.

We shall now prove that this n-fold spherical surface can be transformed without tearing into the surface of a body with p holes in it.

First, suppose we have only two sheets, connected by a single cross-line which joins the branch-points AB. Let the figure (1) represent a section by the plane which bisects AB at right angles.

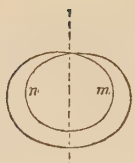


Fig. 1.

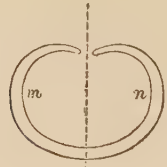


Fig. 2.

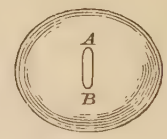


Fig. 3.

Suppose each hemisphere of the inner sheet to be moved across the plane of the great circle containing AB (indicated by the dotted line in the figure), so that the points m, n change places. In this process the two hemispheres will have to penetrate and cross each other; but this may be supposed to take place without altering the continuity of either. Each point may be supposed to move on a straight line perpendicular to the dotted plane, till it coincides with what was its reflexion in regard to that plane. The effect on the cross-line will be to change it from the form drawn in Fig. 1 to that drawn in Fig. 2; instead of the two sheets crossing along the line, each of them will be doubled under it. The result is that, if we now look down on the double sphere from a point vertically over the line AB, we shall see a spherical shell with a hole in it, in the form of a slit along the line AB. Conceive the spherical shell to be made of india-rubber or some more elastic substance; then by mere stretching, without tearing, the slit may be opened out until the shell takes the form of a flat plate; that is, of a body with *no* holes in it.

Next, consider a two-sheeted spherical surface with $p+1$ cross-lines, and suppose them all arranged along the same great circle; which may obviously be done by stretching, without tearing, the surface. Let this great circle be the one represented by the dotted line in Figs. 1 and 2. Then we may apply to the inner sheet the same process as before; viz., we may interchange the two hemispheres into which the sheet is divided by the dotted plane. The effect is to convert all the cross-lines into slits or holes in a spherical shell; and we have supposed that there are $p+1$ of these. One of the slits may be stretched out in the same way as AB was before, so as to convert the spherical shell into a flat plate; but in this flat plate there will remain p holes. A double sphere with $p+1$ crossing lines is thus converted, without tearing, into the surface of a body with p holes in it.

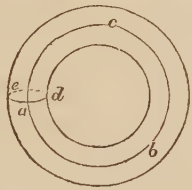
Lastly, suppose that the inner sheet of this two-sheeted sphere is connected by one cross-line with a third inside sheet, the third sheet by one cross-line with a fourth inside it, and so on, until there are n sheets. Let the inner sheet of all be reflected in regard to the plane of the great circle through its crossing line, so that it makes with the

sheet next to it a spherical shell with one hole in it. Then, without tearing, the inner sheet may be shrunk up until it merely covers over this hole. The same process may now be applied to shrink up the second sheet into the third, and so on, until we are left with only the two outside sheets connected by $p+1$ cross-lines. These, however, as we have seen, may be converted, without tearing, into the surface of a body with p holes in it. Hence the proposition follows, that *an n-sheeted Riemann's surface with w branch-points may be transformed, without tearing, into the surface of a body with* p, $= \frac{1}{2}w-n+1$, *holes in it.*

V.

The Number of Irreducible Circuits.

A closed curve drawn on a surface is called a *circuit.* If it is possible to move a circuit continuously on the surface until it shrinks up into a point, the circuit is called *reducible;* otherwise it is *irreducible.* In general there is a finite number of irreducible circuits on a closed surface which are *independent*, that is, no one of which can be made by continuous motion to coincide with a path made out of the others. All other irreducible circuits can then be expressed as compounds of these independent ones. For example, on the surface of a ring (*i. e.*, of a body with one hole through it) there are two independent irreducible circuits; one *round the hole*, as *abc*, and one *through the hole*, as *ade*.



If a circuit goes neither round the hole nor through the hole, it can be shrunk up into a point. If it cannot be so shrunk up, it must go a certain number of times round or through the hole or both, that is, it may be made up of circuits like *abc* and *ade*.

In the same way we may see that, on the surface of a body having p holes through it, there are $2p$ independent irreducible circuits; one *round* each hole, and one *through* each hole. For simplicity consider

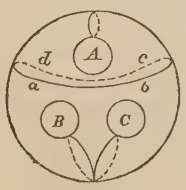


Fig. 1.

Fig. 2.

the case $p=3$. We suppose the body in the form to which we reduced the Riemann's surface, namely, that of a flat plate, represented by Figs. 1 and 2, in which A, B, C are the holes. The circuits through each hole are so drawn as to connect the hole directly with the outer rim, like the circuit which is drawn through the hole A. A circuit passing through *two* holes, as B, C (Fig. 1), may be moved continuously till it consists of two circuits going through the two holes separately. Similarly, a circuit round two or more holes, as B, C (Fig. 2), may be pinched at various points until it is made up of circuits round the separate holes. Such a circuit as *abcd* (Fig. 1) may be moved into the form *abcd* (Fig. 2), in which it consists of two circuits going through the hole A, but in opposite directions. On this account it may be called a *nugatory* circuit.

VI.

The Canonical Dissection.

Suppose now that it is desired to cut through the Riemann's surface in such a way that it shall still hang together, but that it shall no longer be possible to draw an irreducible circuit upon it. This we may do if we successively prevent the different kinds of irreducible circuits considered in the last section. To prevent the possibility of going *round* any hole, we must cut the surface along a circuit which goes *through* the hole. To prevent the passage *through* a hole, we must cut through a circuit which goes *round* a hole.

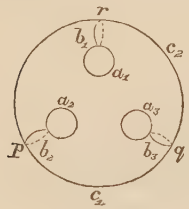


Fig. 3.

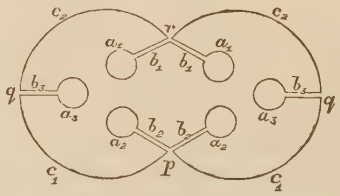


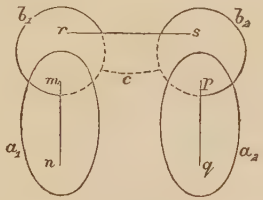
Fig. 4.

Let us make sections a_1, a_2, a_3 round the holes, and b_1, b_2, b_3 through the holes. Then we shall have prevented the drawing of any irreducible circuits except nugatory ones, like *abcd* in the previous figures. To prevent these also, we may cut the surface along the line c_1 which goes from p to q, that is, from a point on b_2 to a point on b_3, and along the line c_2 which goes from q to r, that is, from a point on b_3 to a point on b_1. We must not cut from r to p also, for then we should divide the surface into two separate parts. We may now open out the upper and under portions of the surface in Fig. 3, until it assumes the form of Fig. 4. It then becomes obvious that all our cuts form a continuous

line, which is now the boundary of the surface, and is made up of the pieces (beginning at p and going round to the right) $c_1 b_3 a_3 b_3 c_2 b_1 a_1 b_1 a_1 b_1 c_2 b_3 a_3 b_3 c_1$. Moreover, it is a matter of intuition that no irreducible contour can now be drawn on the surface.

This system of cuts is called a *canonical dissection* of the surface. In the general case it consists of p cuts a going round the holes, p cuts b going through them, and $p-1$ cuts c joining b_2 to b_3, b_3 to b_4, ... b_p to b_1, but not b_1 to b_2. The cuts c may, if we like, join the a-cuts together, or generally they may join the systems (ab) together, a *system* meaning an a-cut and a b-cut belonging to the same hole. In fact, the c-cuts are only of importance as completing the single boundary of the surface, and so enabling us to see that no irreducible circuit is any longer possible.

It only remains to translate this result so that it may be applicable to the original form of the Riemann's surface, viz., an n-fold plane. We shall do this in the case $p=2$, which will sufficiently explain the general case. We have now two sheets connected by three cross-lines



mn, *pq*, *rs*. One of these must be chosen to represent the outer rim of our flat plate; the other two will then correspond to the holes in it. Let *mn*, *pq* represent the holes, and *rs* the outer rim; lines in the upper sheet shall be drawn in full, and lines in the lower sheet shall be dotted. Then we must first make cuts a_1, a_2, which go round the holes *mn*, *pq*; these may lie entirely in the upper sheet. Next we must make cuts b_1, b_2, which connect the holes respectively with the outer rim *rs*. These cuts lie partly in the upper sheet, where they intersect the cuts a, and partly in the under sheet. Lastly, we must connect the system $a_1 b_1$ with the system $a_2 b_2$ by the cut c; this is drawn in the figure from b_1 to b_2 in the under sheet. It is impossible to draw an irreducible circuit on the two-fold plane when it is thus dissected.*

In general, we have proved that in the n-sheeted Riemann's surface which represents the function s determined by the equation

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* It is to be understood that a circuit is *reducible* when all parts of it can be continuously moved away to infinity without crossing any branch-point; because in this theory infinity counts as a single point.

$f(s, z) = 0$, there are $p+1$ cross-lines such that if one be taken to represent the rim, and the rest holes, of a flat plate, the surface may be dissected into one on which no irreducible contour is possible by the following process:—Cut the surface along curves a each of which goes round one of the cross-lines taken to represent holes, on one of the sheets of the surface which cross at that line. Connect each of these lines with the one taken to represent the rim by a cut b along a closed curve which crosses each of the two cross-lines once. Then connect the systems (ab) chainwise by $p-1$ cuts c.

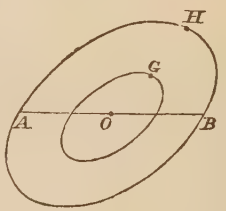
Geometrical Theorems relating to Mean Values.

By Prof. Crofton, F.R.S.

[*Read June 14th*, 1877.]

The following theorems are interesting as examples of the employment of the Theory of Probability to establish mathematical results:—

I. In any convex plane figure *which has a centre* O, let a straight line AOB revolve round O, and trace the locus of the centre of gravity G of either half into which it divides the area; let P be the perimeter of this locus; then $\frac{1}{4}$P *will be equal to the mean distance of the centre* O *from all points in the given area.*



For, let us consider the probability that, if a point be taken at random in the area, and also a straight line cross it at random,* the latter shall pass between the point and the centre O. This may be done in two ways:—

(1) Let the point X be fixed; the chance of the random line pass-

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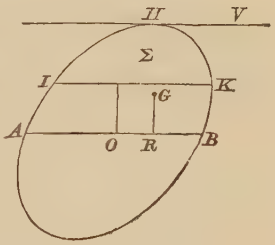
* By this is meant that an infinity of lines are drawn at random in the plane, and one is taken at random from among those which meet the area. This definition is equivalent to that stated by M. Bertrand:—The given area is supposed to be thrown at random on a floor ruled with parallel lines, so distant that it cannot fall on more than one; then the series of chords found by marking on the area the portion of one of the lines within it (whenever it does fall on a line), on each trial out of an infinite number of trials, will be the system of random chords one of which is chosen. (See Bertrand, *Calc. Int.*, p. 487; also *Phil. Trans.*, 1868.)

ing between X and O is $\frac{2OX}{L}$,* L being the perimeter of the figure. Hence, if X ranges all over the area, the required chance is

$$p = \frac{2}{L} \times \text{mean value of OX} = \frac{2M}{L}.$$

(2) Let IK be a fixed position for the random line; the chance of the point X falling on the side away from O is $\frac{\Sigma}{\Omega}$, where Σ is the segment IHK, Ω the given area. Hence the chance is, when the line occupies any of its possible positions,



$$p = \frac{1}{\Omega} \times \text{mean value of } \Sigma,$$

for all positions of IK.

Hence $$\frac{2M}{L} = \frac{1}{\Omega} M(\Sigma).$$

Now if p, ω be the perpendicular from O on IK, and its inclination to a given axis

$$M(\Sigma) = \frac{\iint \Sigma \, dp \, d\omega}{L},$$ †

the integration extending to all positions of the chord IK; Σ always standing for the area of that segment which does not contain O.

To find the integral, let ω be constant, then we have to find $\int \Sigma \, dp$ for all positions of IK between a tangent HV and a diameter AB, both parallel to it; put C = chord IK,

$$\int \Sigma \, dp = p\Sigma - \int p \, d\Sigma,$$

also $$d\Sigma = -C \, dp.$$

Hence, between the above limits,

$$\int \Sigma \, dp = \int p C \, dp = \tfrac{1}{2}\Omega p_1,$$

where $p_1 = GR$, the distance from AB of the centre of gravity of AHB.

Hence $$M = \frac{1}{2\Omega} \iint \Sigma \, dp \, d\omega = \tfrac{1}{4} \int_0^{2\pi} p_1 \, d\omega.$$

Now, as the diameter AB revolves, the tangent at G to the locus of

* It is shewn (*Phil. Trans.*, 1868, p. 185; see also Williamson's *Integral Calculus*, 2nd edition, p. 329) that, if a plane is covered by an infinite number of random lines, the number of lines meeting any convex boundary is measured by L, the perimeter.

† See Williamson, *ub. sup.*

G is always parallel to AB, since, for an infinitesimal rotation of AB, the displacement of G is parallel to the line joining the centre of gravity of the infinitesimal triangle *cut off* from AHB, with that of the equal triangle *added on* to AHB; hence p_1 = perpendicular from O on that tangent, ω = inclination of p_1; hence, by Legendre's theorem, on rectification, $\int_0^{2\pi} p_1 \, d\omega = \mathrm{P}$, the perimeter of that locus;

$$\therefore \quad \mathrm{M} = \tfrac{1}{4}\mathrm{P}.$$

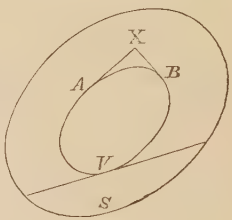
Another proof of this result may be obtained by first proving the following theorem:—

If ρ be the radius of curvature at G of the locus of G,

$$\rho = \frac{\mathrm{AB}^3}{6\Omega}.$$

This result, which easily follows from considering the displacement of G for a small rotation of AB, gives us the *intrinsic* equation of the locus of G, if AB is known as a function of ω.

II. Let there be any two convex boundaries so related that a tangent at any point V to the inner, cuts off a constant segment S from the outer (*e. g.*, two concentric similar ellipses); let the annular area between them be called A; from a point X taken at random on this annulus draw tangents XA, XB to the inner. Find the mean value of the arc AB.



We shall find $\mathrm{M(AB)} = \mathrm{L}\dfrac{\mathrm{S}}{\mathrm{A}}$,

L being the whole length of the inner curve ABV.

We will first prove the following lemma:—

If there be any convex arc AB, and if N_1 be (the measure of) the number of random lines which meet it *once*, N_2 the number which meet it *twice*,

$$2 \text{ arc AB} = \mathrm{N}_1 + 2\mathrm{N}_2.$$

For draw the chord AB, the number of lines meeting the convex figure so formed is

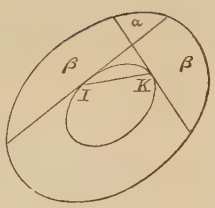
$$\mathrm{N}_1 + \mathrm{N}_2 = \text{arc} + \text{chord (the perimeter)};$$

but N_1 = number of lines meeting the chord = 2 . chord,

$$\therefore \quad 2 \text{ arc} + \mathrm{N}_1 = 2\mathrm{N}_1 + 2\mathrm{N}_2, \quad \therefore \quad 2 \text{ arc} = \mathrm{N}_1 + 2\mathrm{N}_2.$$

Now fix the point X, and draw XA, XB. If a random line cross the boundary L, and p_1 be the probability that it meets the arc AB once, p_2 that it does so twice,

$$\frac{2AB}{L} = p_1 + 2p_2;$$



and if the point X range all over the annulus, and p_1, p_2 are the same probabilities for all positions of X,

$$\frac{2}{L} M(AB) = p_1 + 2p_2.$$

Let now IK be any position of the random line; drawing tangents at I, K, it is easy to see that it will cut the arc AB *twice* when X is in the space marked α, and *once* when X is in either space marked β; hence, for this position of the line,

$$p_1 + 2p_2 = \frac{2\alpha + 2\beta}{A} = \frac{2S}{A}, \text{ which is constant; hence } \frac{M(AB)}{L} = \frac{S}{A}.$$

Hence the mean value of the arc is the same fraction of the perimeter that the constant area S is of the annulus.

If L be not related as above to the outer boundary,

$$\frac{M(AB)}{L} = \frac{M(S)}{A},$$

M(S) being the mean area of the segment cut off by a tangent at a random point on the perimeter L.

The above result may be expressed as an integral. If s be the arc AB included by tangents from any point (x, y) on the annulus,

$$\iint s\, dx\, dy = LS.$$

It has been shewn (*Phil. Trans.*, 1868, p. 191) that, if θ be the angle between the tangents XA, XB,

$$\iint \theta\, dx\, dy = \pi(A - 2S).$$

The mean value of the tangent XA or XB may be shewn to be

$$M(XA) = \frac{S}{2A} P,$$

where P = perimeter of locus of centre of gravity of the segment S. Theorem I. is a case of this.

III. If C be the length of a chord crossing any convex area Ω; Σ, Σ' the areas of the two segments into which it divides the area; and p, ω be the *coordinates* of C, viz., the perpendicular on C from any fixed pole, and the angle made by p with any fixed axis; then

$$\iint C^4\, dp\, d\omega = 6 \iint \Sigma\Sigma'\, dp\, d\omega,$$

both integrations extending to all possible values of p, ω which give a line meeting the area.

This identity will follow by proving that, if ρ be the distance between two points taken at random in the area, the mean value of ρ will be

$$M(\rho) = \frac{1}{\Omega^2} \iint \Sigma\Sigma' dp\, d\omega \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and also

$$M(\rho) = \frac{1}{6\Omega^2} \iint C^4\, dp\, d\omega \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The first follows by considering that, if a random line crosses the area, the chance of its passing between the two points is $\frac{2}{L} M(\rho)$, L being the perimeter of Ω. Again, for any given position of the random line C, the chance of the two points lying on opposite sides of it is $\frac{2\Sigma\Sigma'}{\Omega^2}$; therefore, for all positions of C, the chance is $\frac{2}{\Omega^2} M(\Sigma\Sigma')$; but the mean value $M(\Sigma\Sigma')$, for all positions of the chord, is

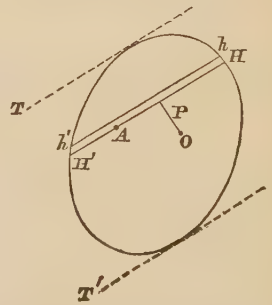
$$M(\Sigma\Sigma') = \frac{\iint \Sigma\Sigma' dp\, d\omega}{\iint dp\, d\omega} = \frac{1}{L} \iint \Sigma\Sigma' dp\, d\omega.$$

To prove equation (2), we remark that the mean value of ρ is found by supposing each of the points A, B to occupy in succession every possible position in the area, and dividing the sum of their distances in each case by the whole number of cases, the measure of which number is Ω^2. Confining our attention to the cases in which the inclination of the distance AB to some fixed direction *lies between* θ *and* $\theta + d\theta$, let the position of A be fixed, and draw through it a chord $HH' = C$, at the inclination θ: the sum of the cases found by giving B all its positions is

$$d\theta \int_0^r \rho \,.\, \rho\, d\rho + d\theta \int_0^{r'} \rho \,.\, \rho\, d\rho = \tfrac{1}{3}(r^3 + r'^3)\, d\theta,$$

where $r = AH$, $r' = AH'$. Now let A occupy successively all positions between HH′ and hh', a chord parallel to it at a distance $= dp$; the sum of all the cases so given will be

$$\tfrac{1}{3} d\theta\, dp \int_0^C (r^3 + r'^3)\, dr = \tfrac{1}{3} d\theta\, dp\, \tfrac{1}{4} 2C^4 = \tfrac{1}{6} d\theta\, dp\, C^4.$$



Now if A moves over the whole area, the sum of the cases will be

$$\tfrac{1}{6} d\theta \int C^4\, dp,$$

where p = perpendicular on C from any fixed pole O, and the integration extends to all parallel positions of C between two tangents T, T′ to the boundary,

the inclination of which is θ. Removing now the restriction as to the direction of the distance AB, and giving it all values from 0 to π, the

sum of all the cases is $\frac{1}{6}\int_0^{\pi} d\theta \int C^4\, dp$;

or, if $\omega =$ inclination of p, $d\omega = d\theta$, and the sum is

$$\tfrac{1}{6}\iint C^4\, dp\, d\omega.$$

The mean value of the *reciprocal* of the distance AB of two points taken at random in a convex area is easily shewn to be

$$M\left(\frac{1}{\rho}\right) = \frac{1}{\Omega^2}\iint C^2\, dp\, d\omega.$$

Thus, for a circle, $M\left(\frac{1}{\rho}\right) = \frac{16}{3\pi r}$.

It may also be shewn that the mean area of the triangle formed by taking three points A, B, C within any convex area is

$$M(ABC) = \Omega - \frac{1}{\Omega^3}\iint C^3\, \Sigma^2\, dp\, d\omega.$$

APPENDIX.

Prof. H. J. S. Smith, in his Presidential Address, frequently refers to the writings of the great German mathematician and physicist, Gauss. The centenary of his birth was celebrated on the 30th April, 1877, at Brunswick. A notice of his life and writings was printed in "Nature" (Vol. xv., No. 390, April 19th), much of the interest of which was due to an analysis of Gauss's powers as a mathematician by Prof. Smith. A list of a few works which were published in connexion with the celebration is given in "Nature" (June 14th, 1877).

Mr. Frankland's paper (p. 57) was printed in "Nature" (April 12th, 1877, No. 389). A letter from Mr. C. J. Monro, questioning some of

the statements in the paper, appeared in No. 391 of the same publication.*

Prof. Smith's paper "On the Conditions of Perpendicularity in a Parallelepipedal System" (p. 83) is given in a slightly different form in the *Philosophical Magazine* (Vol. iv., No. 22, pp. 18–25).

The following results were obtained in Mr. Hammond's paper, "Determinant Conditions for Curves, or Surfaces, of the same Order, having all their Intersections common" (p. 139) :—

Writing in vertical columns the corresponding coefficients of the equations to three curves, or surfaces, of the same degree, a matrix $\left\| \begin{matrix} a_1, & b, & c, & d, & \dots \\ 2 & & & & \\ 3 & & & & \end{matrix} \right\|$ is formed; every determinant of this matrix vanishing when the curves, or surfaces, have all their intersections common.

If all the determinants having two given columns common vanish, all the other determinants of the matrix vanish, and the relations between the coefficients are $(a, b, c) = 0$, $(a, b, d) = 0$, $(a, b, e) = 0$, &c.

Exactly similar results hold for four surfaces having all their inter-

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* Mr. Frankland (formerly of University College and School) is now a resident in New Zealand.

We give a list of works which we have noted as bearing on the subject of Mr. Frankland's paper:—

Gauss, "Werke," Band iv., p. 215 (1828).

Lobatschewsky, N. J., "Principien der Geometrie," Kasan, 1829–30.

Riemann, B., "Ueber die Hypothesen welche der Geometrie zu Grunde liegen," 10 Juli, 1854 ("Abhandlung der K. G. zu Göttingen," Band xiii.).

Lipschitz, "Untersuchungen über die ganzen homogenen Functionen von n Differentialen" (Borchardt, "Journal für Mathematik," Band lxx., 3, 71; lxxiii., 3, 1).

Lipschitz, "Untersuchung eines Problems der Variationsrechnung" (Ibid., Band lxxiv.).

Beltrami, "Saggio di Interpretazione della Geometria non-Euclidea," Napoli, 1868.

Beltrami, "Teoria Fondamentale degli Spazii di Curvatura Costante" ("Annali di Matematica," Serie ii., Tom. ii., pp. 232–255).

Also, translated by Houël, "Annales Scientifiques de l'École Normale," Tom. v., 1869.

Helmholtz, "Ueber die Thatsachen die der Geometrie zum Grunde liegen" ("Nachrichten von der K. G. der Wiss. zu Göttingen," Juni 3, 1868).

Helmholtz, "The Origin and Meaning of Geometrical Axioms" ("Mind," No. iii., pp. 301–321. The above references are taken from this article, as well as from independent reading.)

Helmholtz, "Academy," February 12th, 1870.

Possibly more references may be found in Prof. Clifford's lecture on "The Postulates of the Science of Space," cited by Mr. Frankland; but we have not seen this lecture.

There is a letter in "Nature" (No. 103) on "Helmholtz on the Axioms of Geometry," by Prof. W. S. Jevons.

Schmitz-Dumont, "Zeit und Raum in ihren Denknothwendigen Bestimmungen abgeleitet aus dem Satze des Widerspruchs," I. (Leipzig, 1875. We have not seen any further parts.)

Newcomb, S., "Elementary Theorems relating to the Geometry of a Space of Three Dimensions, and of Uniform Positive Curvature in the Fourth Dimension" ("Journal für Mathematik," Band lxxxiii., Heft 4. The Author styles this a "fairy tale of Geometry.")

sections common, viz., $(a, b, c, d) = 0$, $(a, b, c, e) = 0$, $(a, b, c, f) = 0$, &c., three columns being common to all these determinants.

The number of conditions satisfied by a curve of the n^{th} degree, passing through all the intersections of two other curves of the same degree, is $\frac{(n+1)(n+2)}{2} - 2 = \frac{1}{2}n(n+3) - 1$, which is a well-known theorem in the "Higher Plane Curves."

The number of conditions satisfied by a surface passing through all the intersections of two other surfaces is $\frac{(n+1)(n+2)(n+3)}{6} - 2$, and the number of conditions satisfied by a surface through the intersections of three other surfaces is $\frac{(n+1)(n+2)(n+3)}{6} - 3$.

Mr. Walker's "Equation to the Axes of a Conic" (p. 139) will be found in the "Quarterly Journal of Mathematics," No. 57 (June, 1877, p. 30).

We give here an abstract of Mr. Cotterill's remarks "On a New View of the Pascal Hexagram" (p. 214):—

In a system of coplanar points, the number of intersections of two chords is a multiple of 3. In the case of the hexagram, the 45 points thus derived are divided into four sets of triangles:—1. The three intersections of the chords joining four points form a triad self-conjugate to the conics through the four points. 2. Any three non-conterminous chords intersect in three points forming a diagonal triangle. In each of these two cases, a derived point determines uniquely its corresponding triad, the number of triads being 15. 3. An inscribed triangle determines an opposite inscribed triangle; the three intersections of the pairs of sides supposed to correspond form a triangle the intersections of two inscribed triangles, the nine intersections of the two triangles forming an ennead. 4. The three intersections of the opposite sides of a hexagon of the system form a Pascal triangle. The number of triangles in each of the two last cases is 60, to each triangle of one set corresponding a triangle of the other, as well as a triad of the second set, the nine points forming three triads of the first set. Denoting then the primitive points by italics, and 15 of the derived points (no two of which are conjugate) by Greek letters, we obtain all the derived points by accenting once and twice the Greek letters to form self-conjugate triads. Tables are then formed in matrices of the nine chords joining the vertices of two opposite triangles and their eighteen intersections found to consist of six triangles of each of the second and fourth sets. To these corresponds a matrix containing the nine intersections of the two triangles. In the case of a conic hexagram, the properties of the 60 points of intersection of chords with the tangents at the conic points are then examined.

In the "Philosophical Magazine" for May, 1877 (Vol. iii., No. 19, pp. 360–366), Mr. Muir has a paper, "Extension of a Theorem in Continuants, with an important application" (see p. 215 *antea*).

In the June number of the same publication, Lord Rayleigh has a paper on "Acoustical Observations" (pp. 456–464). This bears on the Note on p. 74 of Vol. vii. of the Society's Proceedings.

The communication on "Some Properties of the Double Theta Functions" (p. 214) was sent by Prof. Cayley to Borchardt's Journal.

A paper, by Mr. Samuel Roberts, bearing on Prof. Cayley's paper on "Three-bar Motion" (Vol. vii., p. 161), and entitled "Geometrical Note on Triangles Inscribed in a Circle and Circumscribed about a Parabola, with reference to the Nodes and Foci of a Three-bar Curve," is printed in No. 57 of the "Quarterly Journal of Mathematics" (p. 52).

A paper by Prof. Rudolf Sturm in the "Mathematische Annalen" (Vol. xii., pp. 254–368), entitled "Ueber Correlative oder reciproke Bündel," as also his paper, "Das Problem der Collineation," bears upon the subject of his communication "On Correlative Pencils" (Vol. vii., pp. 175–194), and upon Dr. Hirst's "Note on the Correlation of two Planes" (pp. 262–273).

Mr. Glaisher has published in the "Messenger of Mathematics" (No. lxxix., Nov. 1877, pp. 102–106) his remarks on "Long Successions of Composite Numbers."

We submit the following abstract of Herr Weichold's paper on "The Irreducible Case," and accompany it with a brief account of its object, for which we are indebted to the kindness of Prof. Cayley :—

Let a, b, c be the roots of the equation

$$x^3 + Ax^2 + Bx + C = 0,$$

and ω, ω' the two complex cubic roots of unity ; and also

$$\rho = a + b\omega + c\omega', \qquad \rho' = a + b\omega' + c\omega,$$
$$\rho_1 = bc + ac\omega + ab\omega', \quad \rho_1' = bc + ac\omega' + ab\omega;$$

$$\text{then } \left.\begin{matrix} a \\ b \\ c \end{matrix}\right\} = \frac{\left.\begin{matrix} -A + \\ \omega \end{matrix}\right\} \begin{matrix} 1 \\ \omega' \\ 1 \end{matrix} \rho + \left.\begin{matrix} 1 \\ \omega \\ \omega' \end{matrix}\right\} \rho'}{3} = -\frac{\left.\begin{matrix} 1 \\ \omega' \\ \omega \end{matrix}\right\} \rho_1 - \left.\begin{matrix} 1 \\ \omega \\ \omega' \end{matrix}\right\} \rho_1'}{\left.\begin{matrix} 1 \\ \omega' \\ \omega \end{matrix}\right\} \rho - \left.\begin{matrix} 1 \\ \omega \\ \omega' \end{matrix}\right\} \rho'}$$

$$= -\frac{\left.\begin{matrix} 2B - \\ \omega \end{matrix}\right\} \begin{matrix} 1 \\ \omega' \\ 1 \end{matrix} \rho_1 - \left.\begin{matrix} 1 \\ \omega \\ \omega' \end{matrix}\right\} \rho_1'}{\left.\begin{matrix} 2A + \\ \omega \end{matrix}\right\} \begin{matrix} 1 \\ \omega' \\ 1 \end{matrix} \rho + \left.\begin{matrix} 1 \\ \omega \\ \omega' \end{matrix}\right\} \rho'} = -\frac{3C}{\left.\begin{matrix} B + \\ \omega \end{matrix}\right\} \begin{matrix} 1 \\ \omega' \\ 1 \end{matrix} \rho_1 + \left.\begin{matrix} 1 \\ \omega \\ \omega \end{matrix}\right\} \rho_1'} \quad \ldots\ldots (a).$$

We have also, after reduction,

$$\begin{aligned}
&\text{(i.)} && \rho\rho' = A^2 - 3B = N,\\
&\text{(ii.)} && \rho_1\rho_1' = B^2 - 3AC = N',\\
&\text{(iii.)} && \rho^3 + \rho'^3 = -2A^3 + 9AB - 27C = M,\\
&\text{(iv.)} && \rho_1^3 + \rho_1'^3 = 2B^3 - 9ABC + 27C^2 = M',\\
&\text{(v.)} && \rho\rho_1' + \rho'\rho_1 = AB - 9C = P,\\
&\text{(vi.)} && \rho\rho_1'^2 + \rho'\rho_1^2 = 9BC - 6A^2C + AB^2 = AN' - 3CN = 2AN' - BP,\\
&\text{(vii.)} && \rho^2\rho_1 + \rho'^2\rho_1' = -9AC + 6B^2 - A^2B = 3N' - BN = AP - 2BN;
\end{aligned}$$

whence $M = 3P - 2AN, \qquad M' = -3CP + 2BN',$

$$\rho\rho_1' - \rho'\rho_1 = \sqrt{P^2 - 4NN'} = S\sqrt{-3}, \quad \rho^3 - \rho'^3 = 3S\sqrt{-3},$$

since $N^2 - A^2N + 3AP = 9N';$

$$\rho^2\rho_1 - \rho_1^2\rho_1' = AS\sqrt{-3}, \quad \rho\rho_1^2 - \rho'\rho_1'^2 = BS\sqrt{-3}, \quad \rho_1^3 - \rho_1'^3 = 3CS\sqrt{-3},$$

since $N'^2 - B^2N' + 3BCP = 9C^2N.$

From the above results we can find

$$\rho^3, \quad \rho'^3, \quad \rho_1^3, \quad \rho_1'^3, \quad \rho\rho_1', \quad \rho'\rho_1, \quad \rho^2\rho_1, \quad \rho'^2\rho_1', \quad \rho\rho_1'^2, \quad \rho'\rho_1'^2 \ldots\ldots (\beta).$$

The values of $\rho, \rho', \rho_1, \rho_1'$ can in effect be found *under finite forms by other elementary operations than the extraction of the cube root*, to wit, *by the decomposition of the last eight quantities in* (β) *into their factors* by means of the determination of "the greatest common divisor."

The determination of the greatest common divisor can be effected by means of a limited number of essentially elementary operations; that is to say, a, b, c are finite algebraical and really performable functions of the coefficients A, B, C of the proposed equation, excepting the case of the three roots being incommensurable all together.

Herr Weichold works out, by his method, the three following equations:

$$x^3 - 16x^2 + 73x - 90 = 0, \qquad x^3 + \frac{41}{140}x^2 - \frac{79}{14}x + \frac{429}{140} = 0,$$

and $$x^3 + \frac{81}{55}x^2 + \frac{181}{385}x + \frac{3}{77} = 0.$$

We take the second of these equations. Here

$$N = \frac{333481}{19600}, \qquad N' = \frac{571333}{19600}, \qquad P = -\frac{57293}{1960},$$

$$S = \frac{380292}{19600}, \qquad \rho\rho_1' = \frac{-286465 + 190146\sqrt{-3}}{19600},$$

$$\rho'\rho_1 = \frac{-286465 - 190146\sqrt{-3}}{19600}.$$

Determination of $\rho, \rho', \rho_1, \rho_1'$ by means of G. C. D. of $\rho\rho_1'$ and $\rho\rho' = N$, for which purpose their C. D., 19600, may for the moment be set aside.

1. $-286465+190146\sqrt{-3} : 333481 = \alpha+\beta\sqrt{-3}$; $\alpha=-1$, $\beta=0$;
remainder, $470146+190146\sqrt{-3}$
$= 6\sqrt{-3}(31691-2612\sqrt{-3})$.

2. $333481 : 31691-2612\sqrt{-3} = \alpha+\beta\sqrt{-3}$; $\alpha=\frac{31691}{3073}$, $\beta=\frac{2612}{3073}$;
nearly $\alpha=10$, $\beta=1$; remainder, $8735-5571\sqrt{-3}$.

3. $31691-2612\sqrt{-3} : 8735-5571\sqrt{-3} = \alpha+\beta\sqrt{-3}$; $\alpha=\frac{961}{508}$,
$\beta=\frac{461}{508}$; nearly $\alpha=2$, $\beta=1$; remainder, $-2492-205\sqrt{-3}$.

4. $8735-5571\sqrt{-3} : -2492-205\sqrt{-3} = \alpha+\beta\sqrt{-3}$; $\alpha=-\frac{55}{19}$,
$\beta=\frac{47}{19}$; nearly $\alpha=-3$, $\beta=2$; remainder, $29-1202\sqrt{-3}$.

5. $-2492-205\sqrt{-3} : 29-1202\sqrt{-3} = \alpha+\beta\sqrt{-3}$; $\alpha=\frac{2}{13}$,
$\beta=-\frac{9}{13}$; nearly $\alpha=0$, $\beta=-1$; remainder, $2(557-88\sqrt{-3})$.

6. $29-1202\sqrt{-3} : 557-88\sqrt{-3} = \alpha+\beta\sqrt{-3}$; $\alpha=1$, $\beta=-2$;

therefore G. C. D. of $-28645+190146\sqrt{-3}$ and 333481 is $557-88\sqrt{-3}$; whence

$$\rho\rho' = \frac{333481}{19600} = \left(\frac{557-88\sqrt{-3}}{140}\right)\left(\frac{557+88\sqrt{-3}}{140}\right)$$
$$= \left(\frac{-557+88\sqrt{-3}}{140}\right)\left(\frac{-557-88\sqrt{-3}}{140}\right);$$

for, as ρ, ρ' are conjugate, it is evident that, when fractional, they must have equal denominators; moreover,

$$\rho\rho_1' = \left(\frac{557-88\sqrt{-3}}{140}\right)\left(\frac{-629+242\sqrt{-3}}{140}\right)$$
$$= \left(\frac{-557+88\sqrt{-3}}{140}\right)\left(\frac{629-242\sqrt{-3}}{140}\right).$$

The comparison of the cubes to

$$\rho^3 = \frac{-133988021+79861320\sqrt{-3}}{2744000},$$

shows that $\rho = \frac{-557+88\sqrt{-3}}{140}$; hence $\rho' = \frac{-557-88\sqrt{-3}}{140}$;

and $\rho_1 = \dfrac{629+242\sqrt{-3}}{140}, \qquad \rho_1' = \dfrac{629-242\sqrt{-3}}{140}.$

Substituting in (α), we get

$$a = -\frac{11}{4}, \quad b = \frac{13}{7}, \quad c = \frac{3}{5}.$$

"The paper is entitled 'Solution of the Irreducible Case,' *i. e.*, of the problem to express the three roots of a complete equation of the third degree, in case of all these roots being real, directly in terms of its coefficients, by means of pure algebraical and really performable operations, whose number shall always be limited except in the case where all these roots are incommensurable.

The problem really considered is that of the solution of a numerical cubic equation with rational coefficients in the case where the three roots, or at least one of them, is rational. It may be remarked, first, that the difficulty is to find one rational root, for, this being known, the equation can be reduced to a quadratic, the roots whereof will be either rational or else will involve quadric radicals only; secondly, that the difficulty arises as well in the case where there is only one real root as in the so-called irreducible case, for in the case of a single real root, which is rational, Cardan's solution does not give this in a rational form; thirdly, that the author's process, although given for the irreducible case only, is perhaps also applicable, or could with some modification be made applicable, to the case of a single real root; and, fourthly, that the idea, although a tolerably obvious one, does seem to present some novelty.

Consider a numerical cubic equation with rational coefficients; take away the second term, and, if necessary, transform the equation so as to multiply the original roots by an integer number m; the equation can be thus brought into the form $x^3+3qx-2r=0$, where q and r are integral; and, solving this by Cardan's method, we have $x=a+b$, and

$$\begin{aligned} a^3+b^3 &= 2r, \\ ab &= -q. \end{aligned}$$

Now, although algebraically these equations give, in regard to a, *nothing* more than do the equations $a^3+b^3=2r$, $a^3b^3=-q^3$, viz., they give $a^3 = r \pm \sqrt{r^2+q^3}$, or $a = \sqrt[3]{r \pm \sqrt{r^2+q^3}}$ (and therefore of course $b = -q \div \sqrt[3]{r \pm \sqrt{r^2+q^3}}$); and although, if writing $r^2+q^3 = \mathrm{M}^2\mathrm{N}$, M the greatest integral square in r^2+q^3, N a positive or negative integer without any square factor, we were from the equation $a^3 = r \pm \mathrm{M}\sqrt{\mathrm{N}}$ to attempt to find r in the form $\alpha+\beta\sqrt{\mathrm{N}}$, we should be simply thrown back upon the original cubic equation; yet arithmetically the equations $a^3 = r \pm \mathrm{M}\sqrt{\mathrm{N}}$, $ab=-q$ do give in regard

to a more than do the equations $a^3+b^3=2r$, $a^3b^3=-q^3$; viz., they give a in the form $\alpha+\beta\sqrt{N}$ as a *common measure* (in the theory of the complex integers of the form $p+q\sqrt{N}$) of the two numbers $r\pm M\sqrt{N}$ and q; and assumiug (as would appear to be allowable) that the numbers α and β are prime to each other,* then they give a in the form in question as the *greatest common measure* (in the theory aforesaid) of the two numbers $r\pm M\sqrt{N}$ and q, such greatest common measure being always calculable by a direct process, at least so long as N is negative and does not exceed a certain numerical limit.

This is the idea of the paper; thus, one of the author's examples is $X^3-16X^2+73X-90=0$ (roots 2, 5, 9). The equation may be reduced to $x^3-111x-110=0$, giving $a^3+b^3=110$, $ab=37$; whence $a^3=55+126\sqrt{-3}$; and a, $=-5+2\sqrt{-3}$, is obtained as the G.C.M. of $55+126\sqrt{-3}$ and 37, in the theory of the complex integers $p+q\sqrt{-3}$."

Mr. S. M. Drach, F.R.A.S., presented to the Society, at the June meeting, the following approximation to π †:—

"In October, 1862, I found and printed in the 'Philosophical Magazine' this simple empirical method to xvi. decimals:—

3000000	2991993 A		
−8007	5983896	$=\frac{1}{5}$A	
20) 2991993 A	35903916		
+14959965	−5983986		
314159265	35897932014		

The left-hand part is superior to 113 : 355; $8007=\frac{3(11^4-6^4)}{11-6}$.

I have since found the co-factor of A to 208 decimals by this continuous process:

	1050000000	1	1998
3	3150000000	3	5994
−8	−8400000	−0	0095984
−7	−7350	−	−83986
	314159265	3	5897932014

the result being $\pi=$A multiplied into this number (periods of 9 digits):

---

* Viz., this will be so, if no unnecessary common factor has been introduced by means of the multiplier m mentioned above.

† He also presented two pages of calculation in connection with the above, to be preserved in the Archives of the Society.

1050000001	199800012	387276101	006754656	243236330
892186519	478803791	421910548	014720956	442088876
993401076	801365540	131037173	765812234	390104048
449219419	767402024	248657640	705154825	615392146
976098979	038819281	438180227	8	

The factor 1238726 is $9\times11\times12\times10427 = 11\left(\frac{9009}{8000} - \frac{9}{10^6}\right)$, and gives result true to xxix. decimals; though I have not succeeded in compressing very greatly the subsequent higher remanets, than by these factors, omitting the precedent cyphers :

$\frac{1}{8} - \frac{2}{8} - \frac{2}{4} + 1101$ ... to xxviii. dec.

$\frac{27}{4} + 48 - 144 - \frac{27}{4} + \frac{1}{4}$ to xxxix. dec.

$-7 - \frac{2}{3} - \frac{7}{3} - 108$ to xlvii. dec.

$+18 + 648\,(100606) + 21$ to lviii. dec.

$\frac{3}{5} + \frac{33}{8} + 168 + 6\times648 + \frac{9}{4} + 48$ to lxxiii. dec.

$14 + 72 + 76 - 36 + 42 + 8$ to lxxxvi. dec.

$100 + 768 + 13 + 65 + 54 + 13 + 104$ to cxii. dec. (if needed).

$-28 - 26 - 234 - 180 - 8$ to cxxii. dec.

$+234 + 39 + 104 + 48 + 45$ to cxxxvi. dec.

$-78 - 5 - 8 - 234 + 14$ to cxlvii. dec.

$202 + 42 + 49 - 343 + 64$ to clix. dec.

$702 + 315 + 48 + 256$ to clxxi. dec.

$-21 + 3888 - \frac{343}{11} + \frac{22}{7} + 96 + 88 - 8$ to ccviii. dec."

R. T.

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	£.	s.	d.
Balance in hand brought forward	15	11	1½
41 Subscriptions, 1875–6	43	1	0
29 do. 1874–5	30	9	0
6 do. 1873–4	6	6	0
2 do. 1872–3	2	2	0
4 do. 1876–7	4	4	0
8 Entrance Fees	8	8	0
January Dividend on Lord Rayleigh's Fund ...	25	0	4
July Dividend on same	21	9	7
January Dividend on Composition Fund	5	6	9
July Dividend on same	5	16	1
Sale of Proceedings and Extra Copies	29	6	0
	£196	19	10½

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Journals	1	17	6
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Attendance	3	7	0
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Book presented by the Council to Mr. Nolet in recognition of his services in repairing models	0	16	6
Proceedings, Printing, &c., paid Messrs. Hodgson & Son, account rendered Nov. 1876	92	16	0
	129	0	4½
Balance in hand	67	19	6
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	£136	4	6

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